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Fully commutative elements and lattice walks

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Abstract. An element of a Coxeter group W is fully commutative if any two of its reduced decompositions are related by a series of transpositions of adjacent commuting generators. These elements were extensively studied by Stembridge in the finite case. In this work we deal with any finite or affine Coxeter group W, and we enumerate fully commutative elements according to their Coxeter length. Our approach consists in encoding these elements by various classes of lattice walks, and we then use recursive decompositions of these walks in order to obtain the desired generating functions. In type A, this reproves a theorem of Barcucci et al.; in type $\widetilde{A}$, it simplifies and refines results of Hanusa and Jones. For all other finite and affine groups, our results are new.

Résumé. Un élément d'un groupe de Coxeter W est dit totalement commutatif si deux de ses décompositions réduites peuvent toujours être reliées par une suite de transpositions de générateurs adjacents qui commutent. Ces éléments ont été étudiés en détail par Stembridge dans le cas où W est fini. Dans ce travail, nous considérons W fini ou affine, et énumérons les éléments totalement commutatifs selon leur longueur de Coxeter. Notre approche consiste à encoder ces éléments par diverses classes de chemins du plan que nous décomposons récursivement pour obtenir les fonctions génératrices voulues. Pour le type A cela redonne un théorème de Barcucci et al.; pour $\widetilde{A}$, cela simplifie et précise des résultats de Hanusa et Jones. Pour tous les autres groupes finis et affines, nos résultats sont nouveaux.

Keywords: Fully commutative elements, Coxeter groups, generating functions, lattice walks, heaps.

Introduction

Let W be a Coxeter group. An element $w \in W$ is said to be *fully commutative* if any reduced expression for w can be obtained from any other one by transposing adjacent pairs of commuting generators. Fully commutative elements were extensively studied by Stembridge in a series of papers [9, 10, 11] where, among others, he classified the Coxeter groups having a finite number of fully commutative elements and enumerated them in each case. In the symmetric group, the fully commutative elements are the 321-avoiding permutations, counted by the Catalan numbers. A nice q-analogue of the Catalan numbers arises when these permutations are enumerated according to their inversion number. This has been done by Barcucci et al. [1], where an elegant expression for the corresponding generating function as a ratio of q-Bessel functions is provided. In the case of the affine symmetric group, a similar q-analogue has been recently found by Hanusa and Jones [7].

The main goal of the present paper is the computation of the generating function $\sum_{w\in W^{FC}} q^{\ell(w)}$, when W is any finite or affine irreducible Coxeter group. Here W^{FC} denotes the subset of fully commutative

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elements of W, and ℓ denotes the Coxeter length. It is known that fully commutative elements in Coxeter groups index a basis for a quotient of the associated Iwahori-Hecke algebra ([4, 5]), so that our formulas give the graded dimensions of these algebras for finite and affine W.

Our investigation in the finite case (Section 2) is based on the *spinal analysis* of Stembridge [9, §2.2], which he uses to find and solve recurrences for the number of elements in W^{FC} when this set is finite. Here we reformulate his results in terms of certain lattice walks (cf. Definition 6), which allows us to take the Coxeter length into account. We then use recursive decompositions of these various families of walks to compute the length generating functions of W^{FC} for all finite W. In type A this gives a simple proof of a result of Barcucci et al. [1]; to our knowledge, the results are new in all other finite types.

In the affine case (Section 3), not tackled by Stembridge, we can also associate lattice walks to fully commutative elements. As in the finite case, decompositions of these walks lead to the computation of the length generating function of W^{FC} for any affine Coxeter group W. The main result of this section is that, in each case, the sequence of coefficients of this generating function is ultimately periodic (Theorem 14). This was already shown in type $\widetilde{A}$ by Hanusa and Jones [7, §5]; however our method gives a much simpler expression for the generating function, and settles positively a question in [7] regarding the beginning of the periodicity. For all other affine types our results are new. In a last section we mention various possible extensions of this work, for instance to involutions.

Finally we point out that the GAP package GBNP was extremely useful at many stages of this work.

1 Fully commutative elements, heaps and walks

1.1 Fully commutative elements in Coxeter groups

Let W be a Coxeter group with finite generating set S, and Coxeter matrix $M = (m_{st})_{s,t\in S}$. That is, M is a symmetric matrix with $m_{ss} = 1$ and, for $s \neq t$, $m_{st} = m_{ts} \in \{2, 3, \ldots\} \cup \{\infty\}$. The relations among the generators are of the form $(st)^{m_{st}} = 1$ if $m_{st} < \infty$. The pair (W, S) is called a Coxeter system. We can write the relations as $sts\cdots = tst\cdots$, each side having length m_{st}; these are usually called *braid relations*; when $m_{st} = 2$, this is a commutation relation. The Dynkin diagram Γ associated to (W, S) is the graph with vertex set S and, for each pair s, t with $m_{st} \geq 3$, an edge between s and t labeled by m_{st} (when $m_{st} = 3$ the edge is left unlabeled).

For $w \in W$ we denote by $\ell(w)$ the minimum length of any expression $w = s_1 \cdots s_l$ with $s_i \in S$. Such expressions with minimum length are called *reduced*, and we denote by $\mathcal{R}(w)$ the set of all reduced expressions of w. A fundamental result in Coxeter group theory is that any expression in $\mathcal{R}(w)$ can be obtained from any other one using only braid relations (see [8]). If an element w satisfies the stronger condition that any reduced expression for w can be obtained from any other one by using only commutation relations, then it is said to be *fully commutative* (which we shall sometimes abbreviate in FC). The following characterization of FC elements is particularly useful.

Proposition 1 ([9], Prop. 2.1) *An element $w \in W$ is fully commutative if and only if for all s, t such that $3 \leq m_{st} < \infty$, there is no expression in $\mathcal{R}(w)$ that contains the factor $\underbrace{sts\cdots}_{m_{st}}$.*

1.2 Heaps

We follow Stembridge [9] in this section. Fix a word $\mathbf{s} = (s_{a_1}, \ldots, s_{a_l})$ in S^*, the free monoid generated by S. We define a partial ordering of the indices $\{1, \ldots, l\}$ by $i \prec j$ if $i < j$ and $m(s_i, s_j) \geq 3$ and extend by transitivity. We denote by $H_{\mathbf{s}}$ this poset together with the “labeling” $i \mapsto s_{a_i}$: this is the *heap* of $\mathbf{s}$. In the Hasse diagram of $H_{\mathbf{s}}$, elements with the same labels will be drawn in the same column. The size $|H|$ of a heap H is the length l of the corresponding word, and for each $s_a \in S$ we let $|H|_a$ be the number of elements in H with label s_a. In Figure 1, we fix a Dynkin diagram on the left, and we give two examples of words with the corresponding heaps.

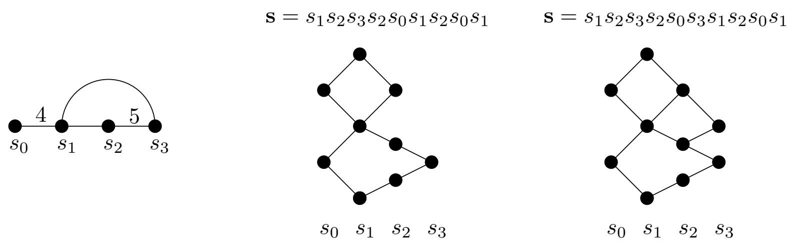


Fig. 1: Two words and their respective heaps.

If we consider heaps up to poset isomorphism which preserve the labeling, then heaps encode precisely *commutativity classes*, that is, if the word $\mathbf{s}'$ is obtainable from $\mathbf{s}$ by transposing commutating generators then there exists a poset isomorphism between $H_{\mathbf{s}}$ and $H_{\mathbf{s}'}$. In particular, if w is fully commutative, the heaps of the reduced words are all isomorphic, and thus we can define the heap of w, denoted by H_w.

A *linear extension* of a poset H is a linear ordering π of H such that $\pi(i) < \pi(j)$ implies $i \prec j$. Now let $\mathcal{L}(H_{\mathbf{s}})$ be the set of words $(s_{\pi(1)}, \ldots, s_{\pi(l)})$ where π goes through all linear extensions of $\mathbf{s}$.

Proposition 2 ([9], Theorem 3.2) *Let w be a fully commutative element. Then $\mathcal{L}(H_{\mathbf{s}})$ is equal to $\mathcal{R}(w)$ for some (equivalently every)* $\mathbf{s} \in \mathcal{R}(w)$.

We say that a chain $i_1 \prec \cdots \prec i_m$ in a poset H is convex if the only elements u satisfying $i_1 \preceq u \preceq i_m$ are the elements i_j of the chain. The next result characterizes *FC heaps*, namely the heaps representing the commutativity classes of FC elements.

Proposition 3 ([9], Proposition 3.3) *A heap H is the heap of some FC element if and only if (a) there is no convex chain $i_1 \prec \cdots \prec i_{m_{st}}$ in H such that $s_{i_1} = s_{i_3} = \cdots = s$ and $s_{i_2} = s_{i_4} = \cdots = t$ where $3 \le m_{st} < \infty$, and (b) there is no covering relation $i \prec j$ in H such that $s_i = s_j$.*

Therefore it is equivalent to characterize FC elements in the Coxeter system (W, S) and heaps verifying the conditions of Proposition 3. The heap on the right of Figure 1 is a FC heap, whereas the one on the left is not since it contains the convex chain with labels (s_2, s_1, s_2) while $m_{s_1s_2} = 3$. In the next section we will exhibit a class of heaps which play an important role for the Coxeter systems we will be interested in.

1.3 Alternating heaps and walks

In all this section, we fix $m_{01}, m_{12}, \ldots, m_{n-1n} \in \{3, 4, \ldots\} \cup \{\infty\}$ and we consider the Coxeter system (W, S) corresponding to the linear Dynkin diagram $\Gamma_n = \Gamma_n((m_{i\,i+1})_i)$ of Figure 2.

Definition 4 (Alternating words and heaps) *A reduced word* $\mathbf{s} \in S^*$ *is alternating if, for $i = 0, \ldots, n-1$, the occurrences of s_i alternate with those of s_{i+1}. A heap is called alternating if it is of the form $H_{\mathbf{s}}$ for an alternating word* $\mathbf{s}$*; a FC element is alternating if its heap is.*

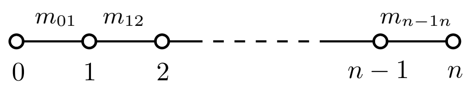


Fig. 2: The linear Dynkin diagram Γ_n.

$\mathbf{s} = 851078627108$

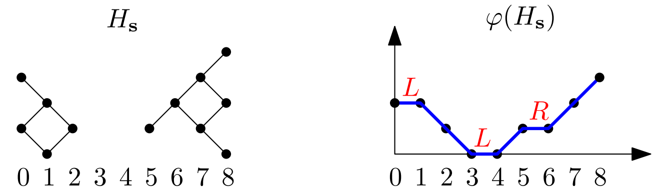


Fig. 3: An alternating word $\mathbf{s}$ with corresponding heap $H_{\mathbf{s}}$, and its encoding by a walk.

An alternating heap is presented in Figure 3. Not all alternating heaps correspond to FC elements, but a characterization is easy by Proposition 3.

Proposition 5 *An alternating heap H is FC if and only if* $\begin{cases} m_{01} > 3 & \textit{or} \quad |H|_0 \leq 1 \\ m_{n-1n} > 3 & \textit{or} \quad |H|_n \leq 1. \end{cases}$

The alternating word/heap in Figure 3 is FC if and only if $m_{01}, m_{n-1n} > 3$.

Definition 6 (Walks) *We call* walk *P of length n a sequence of points $(P_0, P_1, \ldots, P_n)$ in $\mathbb{N}^2$ with n steps in the set $\{(1,1), (1,-1), (1,0)\}$, where the horizontal steps $(1,0)$ are labeled either by L or R. Walks are furthermore required to start at a point P_0 with abscissa 0. A walk is said to satisfy condition $(*)$ if all horizontal steps of the form $(i,0) \to (i+1,0)$ have label L.*

The set of all walks of length n will be denoted by $\mathcal{G}_n$. The subset of walks ending at $P_n = (n,0)$ will be denoted by $\mathcal{Q}_n$, and the subset of $\mathcal{Q}_n$ with $P_0 = (0,i)$ will be denoted by $\mathcal{M}_n^{(i)}$. For short we will write $\mathcal{M}_n = \mathcal{M}_n^{(0)}$ for walks starting and ending on the x-axis. To each family $\mathcal{F}_n \subseteq \mathcal{G}_n$ corresponds subfamilies $\mathcal{F}_n^* \subseteq \mathcal{F}_n$ consisting of those walks in $\mathcal{F}_n$ which satisfy the condition $(*)$, and $\check{\mathcal{F}}_n \subseteq \mathcal{F}_n$ consisting of those walks hit the x-axis at some point.

Remark: Write U, D to represent steps $(1,1), (1,-1)$, and L, R to represent steps $(1,0)$ with these labels; then we can encode our walks by the data of P_0 and a word in $\{U, D, L, R\}$. Consider the injective transformation on such words $U \mapsto UU, D \mapsto DD, L \mapsto UD, R \mapsto DU$. This restricts to a bijection from $\mathcal{M}_n^*$ to Dyck walks, i.e. walks from $(0,0)$ to $(2n,0)$ with steps U, D staying above the x-axis.

The *total height* ht of a walk is the sum of the heights of its points: if $P_i = (i, h_i)$ then $\mathrm{ht}(P) = \sum_{i=0}^{n} h_i$. To each family $\mathcal{F}_n \subseteq \mathcal{G}_n$ we associate the series $F_n(q) = \sum_{P \in \mathcal{F}_n} q^{\mathrm{ht}(P)}$, and we define the generating functions in the variable x by

$$F(x) = \sum_{n \geq 0} F_n(q) x^n \quad \text{and} \quad F^*(x) = \sum_{n \geq 0} F_n^*(q) x^n.$$

We now define a bijective encoding of alternating heaps by walks. In the next section we will exploit this bijection in order to compute some generating functions.

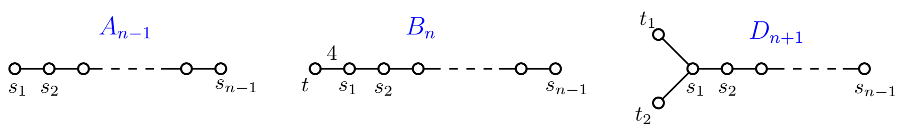
Definition 7 (Map φ) *Let H be an alternating heap of type Γ_n. To each $s_i \in S$ we associate a point $P_i = (i, |H|_i)$. If $|H|_{i+1} = |H|_i > 0$ we label the ith step by L (*resp. *R) if among all elements of H indexed by i and $i+1$, the lowest element has index $i+1$ (*resp. *i). If $|H|_{i+1} = |H|_i = 0$, we label the ith step by L. We define $\varphi(H)$ as the walk $(P_0, P_1, \ldots, P_n)$ with the labels on its steps.*

Theorem 8 *The map $H \mapsto \varphi(H)$ is a bijection between alternating heaps of type Γ_n and $\mathcal{G}_n^*$. The size $|H|$ of the heap is the total height of $\varphi(H)$.*

Proof: The map is clearly well defined since for any alternating heap H, we have $-1 \leq |H|_i - |H|_{i+1} \leq 1$. Fix $(P_0, P_1, \ldots, P_n) \in \mathcal{G}_n^*$. If the step $P_i = (i, h_i) \to P_{i+1} = (i+1, h_{i+1})$ is equal to $(1,1)$ (resp. $(1,-1)$), then we define a convex chain $\mathcal{C}_i$ of length h_{i+1} (resp. h_i) as $(s_{i+1}, s_i, \ldots, s_{i+1})$ (resp. $(s_i, s_{i+1}, \ldots, s_i)$). If the step $P_i \to P_{i+1}$ is labeled by L (resp. R), then we define a chain $\mathcal{C}_i$ of length $h_i = h_{i+1}$ as $(s_i, s_{i+1}, \ldots, s_i, s_{i+1})$ (resp. $(s_{i+1}, s_i, \ldots, s_{i+1}, s_i)$). We define H as the transitive closure of the chains $\mathcal{C}_0, \ldots, \mathcal{C}_{n-1}$. The heap H is uniquely defined, alternating, and satisfies $\varphi(H) = (P_0, P_1, \ldots, P_n)$. Since $h_i = |H|_i$ the result follows. □

2 Finite types

The irreducible Coxeter systems corresponding to finite groups are completely classified ([8]). There are in particular three infinite families whose Dynkin diagrams are given below.



The elements of W^{FC} were enumerated by Stembridge using recurrence relations. Here we reinterpret in many cases the decompositions leading to these recurrences as describing certain lattice walks. The advantage of this approach is that we can take into account the Coxeter length via Theorem 8 and then find recursive decomposions of the walks leading to equations for the generating functions.

For each group W we define $W^{FC}(q) = \sum_{w \in W^{FC}} q^{\ell(w)}$.

2.1 Type A

In this case, we show that it is possible to derive from Theorem 8 the generating function $A^{FC}(x) = \sum_{n\geq 1} A_{n-1}^{FC}(q)x^n$, which was computed in a different way in [1].

Proposition 9 *We have $A_{n-1}^{FC}(q) = M_n^*(q)$, and equivalently $A^{FC}(x) = M^*(x) - 1$.*

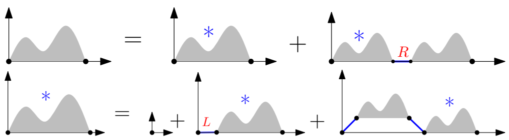
Proof: We are in the setting of Section 1.3 with a diagram Γ_{n-2} and $m_{i,i+1} = 3$ for $i = 0, \ldots, n-3$. The key property in this type is that *all FC elements are alternating* (see Definition 4): this is easy to show (cf. [2, Theorem 2.1]), and is also a particular case of Proposition 15 from Section 3.1. By Proposition 5 and Theorem 8, A_{n-1}^{FC} is then in bijection with walks in $\mathcal{G}_{n-2}^*$ with starting and ending point at height 0 or 1. This set is clearly in bijection with $\mathcal{M}_n^*$: it suffices to add two extra points to $\varphi(H_w)$ on the x-axis (one at the beginning and one at the end). □

Specializing at $q = 1$, this shows that A_{n-1}^{FC} has cardinality the nth Catalan number $Cat_n = \frac{1}{n+1}\binom{2n}{n}$, since $\mathcal{M}_n^*$ is in bijection with Dyck walks by the remark in Section 1.3.

Corollary 10 *The generating function A^{FC} satisfies the following functional equation*

$$A^{FC}(x) = x + xA^{FC}(x) + qxA^{FC}(x)(A^{FC}(qx) + 1). \tag{1}$$

Proof: We have the following walk decompositions with corresponding equations:



$$M(x) = M^*(x) + xM^*(x)M(x), \tag{2}$$

$$M^*(x) = 1 + xM^*(x) + qx^2M^*(x)M(qx). \tag{3}$$

The identity (2) gives $M(x) = M^*(x)/(1-xM^*(x))$, which can be replaced in (3) to yield after some simplifications:

$$M^*(x) = 1 + xM^*(x) + qx(M^*(x) - 1)M^*(qx). \tag{4}$$

Finally, we see through Proposition 9 that (4) is exactly Equation (1). □

Corollary 10 gives another proof of [1, Eq. (3.0.2)], where $A^{FC}(x)$ is denoted by $C(x, q)$. In this work, Barcucci et al. also proved an expression for $C(x, q)$ as a quotient of q-Bessel type functions, using a recursive rewriting rule for 321-avoiding permutations and a result of Bousquet-Mélou [3]. It is possible to derive their ratio of q-Bessel functions by writing

$$A^{FC}(x) + 1 = M^*(x) = \frac{\sum_{n\geq 0} \alpha_n(q)x^n/(x;q)_{n+1}}{\sum_{n\geq 0} \alpha_n(q)x^n/(x;q)_n}, \tag{5}$$

where $(x;q)_n = (1-x)\cdots(1-xq^{n-1})$ stands for the classical q-rising factorial, and $\alpha_n(q)$ is a q-hypergeometric coefficient . Then plugging (5) into (4) yields after a few simplifications and identification $\alpha_n(q) = (-1)^n q^{n(n+1)/2}\alpha_0(q)/(q;q)_n$, as in [1].

2.2 *Type* B

The goal of this subsection is to extend the previous results to the type B. To this aim, we set $B_0^{FC}(q) = 1$, $B_1^{FC}(q) = 1 + q$, and $B^{FC}(x) = \sum_{n\geq 0} B_n^{FC}(q)x^n$. Then we prove the following result.

Proposition 11 *We have* $\quad B^{FC}(x) = Q^*(x) + \dfrac{x^2q^3}{1-xq^2}M^*(x)M(qx).$

Moreover $Q^*(x)$ can be explicitly computed by using the two following equations

$$Q(x) = M(x)(1 + xqQ(qx)) \quad \textit{and} \quad Q^*(x) = M^*(x)(1 + xqQ(qx)). \tag{6}$$

Proof: We first compute the generating function for FC elements in B_n corresponding to alternating heaps of type Γ_{n-1}. By using Proposition 5, we see that these FC elements correspond to alternating heaps H of type Γ_{n-1} with $m_{0,1} = 4$ and $m_{i,i+1} = 3$ for $i = 1, \ldots, n-2$, such that $|H|_{n-1} \leq 1$. Moreover, Theorem 8 implies that $\varphi(H)$ is a path in $\mathcal{G}^*_{n-1}$ with ending point at height 0 or 1. This set is clearly in bijection with $\mathcal{Q}^*_n$: it suffices to add an extra point to $\varphi(H)$ on the x-axis at the end. Adding the trivial FC element corresponding to $n = 0$, one gets the generating function $Q^*(x)$.

It remains to compute the generating function for FC elements in B_n which do not correspond to alternating heaps. Following Stembridge [11, Lemma 2.3], we know that these elements correspond to heaps $H_{\mathbf{s}}$ containing exactly one t, two s_1's, ..., two s_m's (for some $m \in \{1, \ldots, n-1\}$) and such that (a) the restriction of $\mathbf{s}$ to $t, s_1, \ldots, s_m$ is $(s_m, \ldots, s_1, t, s_1, \ldots, s_m)$, and (b) the restriction of $\mathbf{s}$ to $s_m, \ldots, s_{n-1}$, where we also delete one of the two s_m's, is an alternating word. Thanks to Theorem 8 the subwords in (b) are in bijection with $\mathcal{M}_{n-m}^{(1)*}$. This shows that the generating function of the non alternating FC elements in B_n is $M^{(1)*}(x) \times xq^2/(1-xq^2)$. We derive $B^{FC}(x)$ by using the relation $M^{(1)*}(x) = xqM^*(x)M(qx)$, which can easily be obtained by splitting the path at the first intersection with the x-axis. Similar decompositions finally yield the equations (6). $\square$

Note that the elements in B_n^{FC} corresponding to alternating heaps are called *fully commutative top elements of* B_n in [10], and *commutative elements of the Weyl group* $\mathcal{C}_n$ in [4]. Therefore $Q^*(x)$ gives a generating function for these particular elements.

2.3 Type D

In this subsection, we will see how it is possible to derive from the previous results in type B analogous expressions in type D. We set $D_1^{FC}(q) = 1$, $D_2^{FC}(q) = (1+q)^2$, and $D^{FC}(x) = \sum_{n \geq 0} D_{n+1}^{FC}(q)x^n$, and we prove the following result.

Proposition 12 *We have* $$D^{FC}(x) = 2Q^*(x) - M^*(x) + \frac{xq^2}{1-xq^2}M^*(x)M(qx).$$

Proof: Reformulating Stembridge [11, §3.3], each element of D_{n+1}^{FC} can be obtained from one of B_n^{FC} by exactly one of the following rules. Consider an alternating element w in B_n^{FC} and $\mathbf{s} \in \mathcal{R}(w)$. If $\mathbf{s}$ contains no occurrence of t it yields an element of D_{n+1}^{FC}. If $\mathbf{s}$ contains at least one occurrence of t, then we can replace its subword $(t, \ldots, t)$ by $(t_1, t_2, \ldots)$ or $(t_2, t_1, \ldots)$, giving rise to two elements in D_{n+1}^{FC}. If $\mathbf{s}$ contains exactly one occurrence of t, we obtain a FC element of type D by replacing t by t_1t_2 $(= t_2t_1)$. By the proof of Proposition 11, the generating function corresponding to these three families is given by

$$M^*(x) + 2(Q^*(x) - M^*(x)) + qM^{(1)*}(x).$$

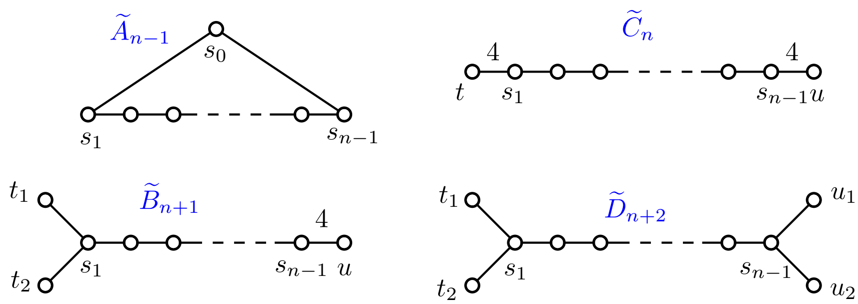
The remaining elements of D_{n+1}^{FC} are simply obtained from non alternating elements of B_n^{FC} by again replacing t by t_1t_2, therefore yielding $M^{(1)*}(x) \times xq^3/(1-xq^2)$, and the results follows. $\square$

2.4 The exceptional cases

The exceptional types are $I_2(m), H_3, H_4, F_4, E_6, E_7$, and E_8. For the dihedral group $W = I_2(m)$ one easily has $I_2(m)^{FC}(q) = 1 + 2q + 2q^2 + \cdots + 2q^{m-1} = 1 + 2q(1-q^{m-1})/(1-q)$. For the remaining cases, we used the computer to find the generating polynomials $W^{FC}(q)$; for instance, $E_8(q)$ has degree 29 in q. Note that the number of FC elements in a Coxeter group may be finite even though the group itself is infinite: Stembridge [9] discovered that there are three families $E_n(n > 8), F_n(n > 4), H_n(n > 4)$ of infinite groups with a finite number of FC elements. Extending Stembridge [11] with similar walk techniques, it is possible to enumerate such elements according to their length.

3 Affine types

The Dynkin diagrams of the infinite families of affine groups are represented below.



Lemma 13 *Suppose $F(q) = \sum_{i\geq 0} a_i q^i = P(q)/(1-q^N)$ where $P(q)$ is a polynomial of degree d. Then one has $a_{i+N} = a_i$ for all $i \geq d$. Furthermore the average value over a period $(a_i + a_{i+1} + \cdots + a_{i+N-1})/N$ is equal to $P(1)/N$ for $i \geq d$.*

We will show that this lemma applies to all generating functions $W^{FC}(q)$ when W is affine.

Theorem 14 *For each irreducible affine group W, the sequence of coefficients of $W^{FC}(q)$ is ultimately periodic, with period recorded in the following table ($\widetilde{F}_4^{FC}, \widetilde{E}_8^{FC}$ are finite sets):*

AFFINE TYPE	$\widetilde{A}_{n-1}$	$\widetilde{C}_n$	$\widetilde{B}_{n+1}$	$\widetilde{D}_{n+2}$	$\widetilde{E}_6$	$\widetilde{E}_7$	$\widetilde{G}_2$	$\widetilde{F}_4, \widetilde{E}_8$
PERIODICITY	n	$n+1$	$(n+1)(2n+1)$	$n+1$	4	9	5	1

The periods for $\widetilde{B}_{n+1}$ look surprising at first sight, but they are experimentally close to the actual minimal periods: for $n+1 = 3, 4, 5, 6$, the formula gives $15, 28, 45, 66$ while the minimal periods are $15, 7, 45, 33$.

The proof of Theorem 14 will be detailed in type $\widetilde{A}_n$, and simply outlined for other types.

3.1 Type $\widetilde{A}_n$

The generating function $\widetilde{A}_{n-1}^{FC}(q)$ was computed by Hanusa and Jones [7]; we will compare our results to theirs at the end of this section.

Proposition 15 *An element $w \in \widetilde{A}_{n-1}$ is fully commutative if and only if, in any reduced decomposition of w, the occurrences of s_i and s_{i+1} alternate for all $i \in \{0, \ldots, n-1\}$, where we set $s_n = s_0$.*

Proof: The condition is clearly sufficient by using Proposition 1. To show that it is necessary, assume $w \in \widetilde{A}_{n-1}^{FC}$ and $w_1 \cdots w_l$ is a reduced word for w ($w_k \in S$). We will show that between two consecutive s_i there is necessarily a s_{i+1}, which suffices to prove the proposition. Assume then for the sake of contradiction that there exist $i \in \{0, \ldots, n-1\}$ and $j_1 < j_2$ such that $w_{j_1} = w_{j_2} = s_i$ and that for all j satisfying $j_1 < j < j_2$ one has $w_j \neq s_i, s_{i+1}$. Among all possible i, j_1, j_2, pick one with $j_2 - j_1$ minimal. Now consider the number m of indices j with $j_1 < j < j_2$ and $w_j = s_{i-1}$. If $m = 0$, then by successive commutations we see that the word is not reduced, which is excluded. If $m = 1$, then by successive commutations one obtains a factor $s_i s_{i-1} s_i$ which is excluded by Proposition 1. If $m \geq 2$, then two consecutive occurrences of s_{i-1} contradict the minimality of $j_2 - j_1$. This finishes the proof. □

We are not exactly in the case of Section 1.3 since the Dynkin diagram is not linear, but one can nonetheless define walks from the "alternating" words described in Proposition 15: given $w \in \widetilde{A}_{n-1}^{FC}$ and $i = 0, \dots, n-1$, draw a step from $P_i = (i, |w|_{s_i})$ to $P_{i+1} = (i+1, |w|_{s_{i+1}})$ as in the definition of φ; here $|w|_s$ is the number of occurrences of s in any reduced decomposition of w. This forms a path noted $\varphi'(w)$ of length n, with both P_0 and P_n at height $|w|_{s_0}$.

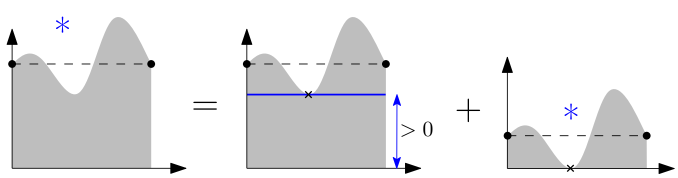
So we have $\varphi'(w) \in \mathcal{O}_n$, defined as the set of paths P in $\mathcal{G}_n$ whose starting and ending point are at the same height. Define also $\mathrm{ht}'(P) = \sum_{i<n} h_i$ where $P_i = (i, h_i)$, so that we count just once the height of the final and initial point (one should think of these walks as being on a cylinder, with these two points coinciding). We also define $O_n(q)$ to be the generating function of $\mathcal{O}_n$ with respect to the modified total height ht'. Finally, denote by $\mathcal{E}_n$ the set of horizontal walks in $\mathcal{O}_n^*$ with all vertices at height $h > 0$ and all steps with the same label L or R.

Theorem 16 *The map $\varphi' : \widetilde{A}_{n-1}^{FC} \to \mathcal{O}_n^* \setminus \mathcal{E}_n$ is a bijection such that $\ell(w) = \mathrm{ht}'(\varphi'(w))$.*

Proof (Sketch): The walks in $\mathcal{E}_n$ are not of the form $\varphi'(w)$: indeed, given an element $w \in \widetilde{A}_{n-1}^{FC}$ and any reduced word for it, consider the positions $j_0, j_1, \dots, j_{n-1}$ of the leftmost $s_0, s_1, \dots, s_{n-1}$ respectively. Suppose $\varphi'(w)$ consists of horizontal steps labeled R: then this would imply $j_0 < j_1 < \cdots < j_{n-1} < j_0$ which is a contradiction. The case of labels L is similar, so the image of φ' is included in $\mathcal{O}_n^* \setminus \mathcal{E}_n$. The function φ' is injective, since, as for Theorem 8, the walk $\varphi'(w)$ allows us to reconstruct the heap H_w.

For the surjectivity we omit the proof in this abstract: one has to check that when trying to reconstruct the possible heap of w from a walk $P \in \mathcal{O}_n^* \setminus \mathcal{E}_n$ as in the proof of Theorem 8, one always gets the Hasse diagram of a poset in the end. $\square$

As an immediate corollary we have that $\widetilde{A}_{n-1}^{FC}(q) = O_n^*(q) - 2q^n/(1-q^n)$. Now we have to count walks in $\mathcal{O}_n^*$, and to this end we decompose them according to their lowest point: this is pictured below and gives the equation $O_n^*(q) = \check{O}_n^*(q) + q^n \check{O}_n(q)/(1-q^n)$.

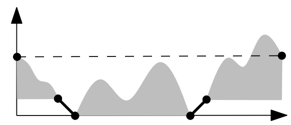


By Theorem 16, we thus have the generating function

$$\widetilde{A}_{n-1}^{FC}(q) = O_n^*(q) - 2\frac{q^n}{1-q^n} = \frac{q^n(\check{O}_n(q) - 2)}{1-q^n} + \check{O}_n^*(q). \tag{7}$$

Note that $\check{O}_n^*(q)$ and $\check{O}_n(q)$ are *polynomials*, both of degree $\lceil n/2 \rceil \lfloor n/2 \rfloor$. By Lemma 13, the coefficients of $\widetilde{A}_{n-1}^{FC}(q)$ are periodic of period n, and the average value over a period is $\frac{1}{n}\left(\binom{2n}{n} - 2\right)$. Indeed we have $\check{O}_n(1) = \binom{2n}{n}$: to see this, shift any path from $\check{\mathcal{O}}_n$ so that it starts at the origin, and use the transformations $U \mapsto UU, D \mapsto DD, L \mapsto UD, R \mapsto DU$ defined after Definition 6. This is a bijection from $\check{\mathcal{O}}_n$ to paths from the origin to $(2n, 0)$ using steps U or D, and there are obviously $\binom{2n}{n}$ such paths.

We still have to explain how to compute $\check{O}_n(q)$ and $\check{O}^*_n(q)$. Walks in $\check{\mathcal{O}}_n$ (*resp.* in $\check{\mathcal{O}}^*_n$) either belong to $\mathcal{M}_n$ (*resp.* $\mathcal{M}^*_n$) or can be decomposed as in the picture on the right. Using standard techniques, this translates into the equations:



$$\check{O}(x) = M(x)\left(1 + qx^2\frac{\partial(xM)}{\partial x}(xq)\right) \qquad \text{and} \qquad \check{O}^*(x) = M^*(x)\left(1 + qx^2\frac{\partial(xM)}{\partial x}(xq)\right), \tag{8}$$

which allows us to compute easily the generating functions $\widetilde{A}^{FC}_{n-1}(q)$.

Link with [7]: Hanusa and Jones [7] compute the same generating function $\widetilde{A}^{FC}_{n-1}(q)$ by using the realization of $\widetilde{A}_{n-1}$ as the affine symmetric group $\widetilde{S}_n$. It is known [6] that FC elements in this representation correspond to 321-avoiding permutations, extending the finite case. The expressions in [7] are much more complicated than ours, and more difficult to derive. The authors show that periodicity of the coefficients start at the index $2\lfloor n/2\rfloor\lceil n/2\rceil$ but conjecture that $1 + \lfloor(n-1)/2)\rfloor\lceil(n-1)/2\rceil$ is the right beginning. Lemma 13 applied to (7) gives the slightly worse result of $n + \lfloor n/2\rfloor\lceil n/2\rceil$, but we can actually prove their conjecture. Consider the operation Up on $\mathcal{O}^*_n$ which simply adds 1 to the height of each point: this sends a walk P with $\mathrm{ht}'(P) = k$ to a walk Q with $\mathrm{ht}'(Q) = n + k$. Now it is easy to see that if $k > \lfloor(n-1)/2)\rfloor\lceil(n-1)/2\rceil$, then no path P with $\mathrm{ht}'(P) = k$ has an horizontal step at height zero. From this one deduces that if $k > \lfloor(n-1)/2)\rfloor\lceil(n-1)/2\rceil$, then Up is a bijection between paths P with $\mathrm{ht}'(P) = k$ and paths P with $\mathrm{ht}'(P) = n + k$, which proves the conjecture of [7].

3.2 Other affine types

Type $\widetilde{C}_n$: Starting from a certain length (depending on n), we can show that FC elements can be classified in two families: (a) alternating elements, which correspond to walks in $\mathcal{G}^*_n$ by Proposition 3 and the bijection φ from Theorem 8, and (b) exceptional elements, whose reduced words appear as factors of the infinite periodic word $(ts_1s_2\cdots s_{n-1}us_{n-1}\cdots s_2s_1)^\infty$. The heaps of the words in (b) are totally ordered, and for any length $> n+1$ these words are not alternating and there $2n$ of them. Using a decomposition of $\mathcal{G}^*_n$ extending the one for $\mathcal{O}^*_n$, we can then write

$$\widetilde{C}^{FC}_n(q) = \frac{q^{n+1}\check{G}_n(q)}{1 - q^{n+1}} + \frac{2nq^{n+2}}{1-q} + R_n(q), \tag{9}$$

where $R_n(q)$ is a certain polynomial that we are able to compute, and this also holds for $\check{G}_n(q)$ since $\check{G}(x) = M(x)\left(1 + qxQ(qx)\right)^2$ by extending the decomposition we used for $\check{O}(x)$.

Applying Lemma 13 to (9), we have that the coefficients of $\widetilde{C}^{FC}_n(q)$ are ultimately periodic of period $n+1$, and that the average value over a period is $2n + 4^n/(n+1)$. Indeed, $\check{G}_n(1) = 4^n$, which can be immediately seen by shifting down walks of $\check{\mathcal{G}}_n$ so that they start at the origin.

Types $\widetilde{B}_{n+1}$ and $\widetilde{D}_{n+2}$: For a large enough length (depending on n), there are for both types only two families of FC elements (as in type $\widetilde{C}_n$):

(a) Those coming from alternating heaps of type $\widetilde{C}_n$, as in Section 2.3 relating B_n and D_{n+1}. In type $\widetilde{B}_{n+1}$, apply the replacements $(t, t, t, \ldots) \mapsto (t_1, t_2, t_1, \ldots)$ or $(t_2, t_1, t_2, \ldots)$. In type $\widetilde{D}_{n+2}$, apply this map together with the same one with u, u_1, u_2 instead of t, t_1, t_2. The length generating function of these FC elements is clearly ultimately periodic with period $n+1$ in both types $\widetilde{B}_{n+1}$ and $\widetilde{D}_{n+2}$.

(b) Exceptional elements similar to those of type $\widetilde{C}_n$. In type $\widetilde{B}_{n+1}$, these have for reduced words the factors of $(t_1t_2s_1s_2\dots s_{n-1}us_{n-1}\dots s_2s_1)^\infty$ where t_1,t_2 are allowed to commute. For a large enough length i, there are $2n+3$ such elements unless i is divisible by $2n+1$, in which case there are $2n+4$ such factors. In type $\widetilde{D}_{n+2}$, these exceptional elements have for reduced words the factors of $(t_1t_2s_1s_2\dots s_{n-1}u_1u_2s_{n-1}\dots s_2s_1)^\infty$ where u_1,u_2 are also allowed to commute. For a large enough length i, there are $2n+6$ such factors unless i is divisible by $n+1$, in which case there are $2n+8$ factors.

From this, it follows that the coefficients of $\widetilde{B}_{n+1}^{FC}(q)$ (*resp.* $\widetilde{D}_n^{FC}(q)$) are ultimately periodic with period $(n+1)(2n+1)$ (*resp.* $n+1$).

Exceptional types: The remaining irreducible affine types are $\widetilde{E}_6, \widetilde{E}_7, \widetilde{E}_8, \widetilde{F}_4$, and $\widetilde{G}_2$. The number of FC elements in $\widetilde{F}_4$ or $\widetilde{E}_8$ is actually finite, since they correspond in Stembridge's classification [9, Theorem 5.1] to types F_5 and E_9: the corresponding polynomials $\widetilde{F}_4(q)$ and $\widetilde{E}_8(q)$ have degrees respectively 18 and 44. The number of FC elements according to the length is ultimately periodic in the remaining types $\widetilde{E}_6, \widetilde{E}_7, \widetilde{G}_2$, the periods being $4, 9$ and 5 respectively: this is easily seen in type $\widetilde{G}_2$ and requires a finer analysis in the two other types. In all three cases we can actually compute the whole series $W^{FC}(q)$.

4 Further questions

Involutions: As explained by Stembridge in [11], a FC element w in a Coxeter group is an involution if and only if $\mathcal{R}(w)$ is *palindromic*, meaning that it includes the mirror image of some (equivalently, all) of its members. Thanks to Theorem 8, the alternating FC involutions correspond to Motzkin type walks having no horizontal steps at height greater than zero. Therefore, denoting by $\overline{A}^{FC}(x)=\sum_{n\geq 1}\overline{A}_{n-1}^{FC}(q)x^n$, $\overline{B}^{FC}(x)=\sum_{n\geq 0}\overline{B}_n^{FC}(q)x^n$ and $\overline{D}^{FC}(x)=\sum_{n\geq 0}\overline{D}_{n+1}^{FC}(q)x^n$ the generating functions for FC involutions in types A, B and D, we can prove that:

$$\begin{aligned}
\overline{A}^{FC}(x) &= \frac{\mathrm{Cat}(x)}{1-x\mathrm{Cat}(x)}-1,\\
\overline{B}^{FC}(x) &= \frac{\mathrm{Cat}^+(x)}{1-x\mathrm{Cat}(x)}+\frac{x^2q^3}{1-xq^2}\frac{\mathrm{Cat}(x)\mathrm{Cat}(qx)}{1-x\mathrm{Cat}(x)},\\
\overline{D}^{FC}(x) &= 2\cdot\frac{\mathrm{Cat}^o(x)}{1-x\mathrm{Cat}(x)}+\frac{\mathrm{Cat}}{1-x\mathrm{Cat}(x)}+\frac{xq^2}{1-xq^2}\frac{\mathrm{Cat}(x)\mathrm{Cat}(qx)}{1-x\mathrm{Cat}(x)},
\end{aligned}$$

where $\mathrm{Cat}(x)$ (*resp.* $\mathrm{Cat}^+(x)$, *resp.* $\mathrm{Cat}^o(x)$) denotes the generating function for q weighted Dyck walks. (*resp.* suffixes of Dyck walks, *resp.* suffixes of Dyck walks starting at an odd height). Note that all these generating functions can be computed by the functional equations $\mathrm{Cat}(x)=1+qx^2\mathrm{Cat}(x)\mathrm{Cat}(qx)$, $\mathrm{Cat}^+(x)=\mathrm{Cat}(x)(1+xq\mathrm{Cat}^+(qx))$, and $\mathrm{Cat}^o(x)=xq\mathrm{Cat}(x)\mathrm{Cat}(qx)(1+xq^2\mathrm{Cat}^o(q^2x))$.

Other problems: One may try to find a formula for $Q^*(x)$, maybe in terms of certain q-Bessel functions as for $M^*(x)$, since this would give formulas for $B^{FC}(x)$ and $D^{FC}(x)$.

It would be interesting to study other statistics on the sets W^{FC}; for instance, the number of descents, which has the advantage of being defined for any Coxeter group.

In the hyperplane arrangement associated to an affine group W , elements correspond to the regions (named *alcoves*). One may wonder where the alcoves corresponding to FC elements are located, and how the periodicity of Theorem 14 is involved.

Finally, a natural problem is to determine which Coxeter groups W are such that $W^{FC}(q)$ has ultimately periodic coefficients. This is work in progress: there are apparently exactly two exceptional, non-affine groups with such a periodicity.

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