Matroids over a ring

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Abstract. We introduce the notion of a matroid M over a commutative ring R, assigning to every subset of the ground set an R-module according to some axioms. When R is a field, we recover matroids. When $R = \mathbb{Z}$, and when R is a DVR, we get (structures which contain all the data of) quasi-arithmetic matroids, and valuated matroids, respectively. More generally, whenever R is a Dedekind domain, we extend the usual properties and operations holding for matroids (e.g., duality), and we compute the Tutte-Grothendieck group of matroids over R.

Résumé. Nous introduisons la notion de matroïde M sur un anneau commutatif R, qui assigne à chaque partie d'un ensemble E un R-module selon certains axiomes. Quand R est un corps, on retrouve les matroïdes. Lorsque $R = \mathbb{Z}$, et lorsque R est un anneau de valuation discrète, nous obtenons (structures qui contiennent toutes les données) respectivement des matroïdes quasi-arithmétiques et des matroïdes valués. En plus de généralité, quand R est un anneau de Dedekind, nous étendons les propriétés et operations habituelles pour les matroïdes (par exemple, la dualité), et nous calculons le groupe de Tutte-Grothendieck des matroïdes sur R.

Keywords: arithmetic matroids, valuated matroids, tropical flag variety, Dedekind domains, Tutte polynomial, Tutte-Grothendieck group

1 Introduction

The notion of a *matroid* axiomatizes the linear algebra of a list of vectors. Matroid theory has proved to be a versatile language to deal with many problems on the interfaces of combinatorics and algebra. In the years since 1935, when Whitney first introduced matroids, a number of enriched variants thereof have arisen, among them oriented matroids [2], valuated matroids [8], complex matroids [1], and (quasi-) arithmetic matroids [13, 5]. Each of these structures retains some information about a vector configuration, or an equivalent object, which is richer than the purely linear algebraic information that matroids retain.

As a running motivating example, let us focus on quasi-arithmetic matroids. A quasi-arithmetic matroid endows a matroid with a multiplicity function, whose values (in the representable case) are the cardinalities of certain finite abelian groups, namely, the torsion parts of the quotients of an ambient lattice $\mathbb{Z}^n$ by the sublattices spanned by subsets of vectors. From a list of vectors with integer coordinates one may produce objects like a toric arrangement, a partition function, and a zonotope. In order to have a combinatorial structure from which these objects may be read off, one needs to keep track of arithmetic properties of the vectors, and this is what quasi-arithmetic matroids provide.

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It is natural to ask how well these generalizations of matroids can be unified under one framework. Such a unification was sought by Dress in his program of matroids with coefficients, represented for example in his work with Wenzel [8] wherein valuated matroids are matroids with coefficients in a "fuzzy ring".

In the present paper we suggest a different approach to such unification, by defining the notion of a *matroid M over a commutative ring R*. Such an M assigns, to every subset A of a ground set, a finitely generated R-module $M(A)$ according to some axioms (Definition 2.1). We find this definition to have multiple agreeable features. For one, by building on the well-studied setting of modules over commutative rings, we get a theory where the considerable power and development of commutative algebra can be easily brought to bear. For another, unlike arithmetic and valuated matroids, a matroid over R is not defined as a matroid decorated with extra data; there are only two axioms, and we suggest that they are comparably simple to the matroid axioms themselves. In particular, a *representable* matroid over R is precisely a vector configuration in a finitely generated R-module.

When R is a field, a matroid M over R is nothing but a matroid: the data $M(A)$ is a vector space, which contains only the information of its dimension, and this directly encodes the rank function of M. When $R = \mathbb{Z}$, every module $M(A)$ is an abelian group, and by extracting its torsion subgroup we get a quasi-arithmetic matroid. When R is a discrete valuation ring (DVR), we may similarly extract a valuated matroid. More generally, whenever R is a Dedekind domain, we can extend the usual properties and operations holding for matroids, such as duality.

The idea of matroids over rings was suggested by features of the theory of quasi-arithmetic matroids. Some significant information about an integer vector configuration is not retained in the multiplicity function, as many finite abelian groups can have the same cardinality. Recording the whole structure of these groups is desirable in several situations, for example, in developing a combinatorial intersection theory for the arrangements of subtori arising as characteristic varieties. The properties of the multiplicity function of a quasi-arithmetic matroid turn out to be just shadows of group-theoretic properties.

One of the most-loved invariants of matroids is their Tutte polynomial $\mathbf{T}_M(x, y)$. It thus comes as no surprise that the Tutte polynomial has been considered for generalizations of matroids as well. A quasi-arithmetic matroid $\hat{M}$ has an associated arithmetic Tutte polynomial $\mathbf{M}_{\hat{M}}(x, y)$, which has proved to be a useful tool in studying toric arrangements, partition functions, zonotopes, and graphs ([13, 7, 3]). More strongly, the authors of [3] define a *Tutte quasi-polynomial* of an integer vector configuration, interpolating between $\mathbf{T}_M(x, y)$ and $\mathbf{M}_{\hat{M}}(x, y)$, which is no longer an invariant of the quasi-arithmetic matroid (as it depends on the groups, not just their cardinalities).

Among its properties, the Tutte polynomial of a classical matroid is the universal deletion-contraction invariant. In more algebraic language, following [4], the class of a matroid in the Tutte-Grothendieck group for deletion-contraction relations is exactly its Tutte polynomial. While the arithmetic Tutte polynomial and Tutte quasi-polynomial are deletion-contraction invariants, neither is universal for this property. Our generalization of the Tutte polynomial for matroids over a Dedekind ring R is also the class in the Tutte-Grothendieck group, so it retains the universality of the usual Tutte polynomial, and we obtain the two generalizations of Tutte just mentioned as evaluations of it.

This paper is organized as follows. In Section 2 we give the basic definitions for matroids over a commutative ring, including representability, and we explain how they generalize the classical ones.

We introduce the assumption that R is a Dedekind domain, and do some groundwork, in Section 3. This assumption on R remains for the most part in force from this section onward. Its first application comes in Section 4, where we establish the existence and the properties of the dual of a matroid over a Dedekind domain R.

In Section 5 we develop the local theory, with a structure theorem for matroids over a DVR. We show connections with the Hall algebra and with the tropical Plücker relations for the flag variety. Finally, we describe how to recover valuated matroids.

The global theory is developed in Section 6. We describe the structure of a matroid over a Dedekind ring R in terms of the structure of all its localizations (completely described in the previous section) plus some global information coming from the Picard group of R . This also explains the connection between matroids over $\mathbb{Z}$ and quasi-arithmetic matroids.

Finally, in Section 7 we compute the Tutte-Grothendieck group. In particular, given a matroid over $\mathbb{Z}$, we present its Tutte quasi-polynomial as an evaluation of its class in $K(\mathbb{Z}\text{-Mat})$.

This paper is an extended abstract of the article [10], to which the interested reader is suggested to refer for many details and for all the proofs, which are omitted here.

2 Matroids over a ring

By R-Mod we mean the category of finitely generated R-modules over a commutative ring R. We write "f.g." for "finitely generated" throughout.

Definition 2.1 *Let R be a commutative ring. A* matroid over R *on the ground set E is a function M assigning to each subset $A \subseteq E$ a finitely-generated R-module $M(A)$ satisfying the following axioms:*

(M1) For any $A \subseteq E$ and $b \in E \setminus A$, there exists a surjection $M(A) \twoheadrightarrow M(A \cup \{b\})$ whose kernel is a cyclic submodule of $M(A)$.

(M2) For any $A \subseteq E$ and $b \neq c \in E \setminus A$, there exists a pushout

$$\begin{tikzcd} M(A) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & M(A \cup \{b\}) \arrow[d] \\ M(A \cup \{c\}) \arrow[r] & M(A \cup \{b, c\}) \end{tikzcd}$$

where all four morphisms are surjections with cyclic kernel.

Polymatroids can be defined similarly (see [10, Definition 2.2]). Clearly, Axiom (M1) is redundant if $|E| \geq 2$, and the choice of the modules $M(A)$ is only relevant up to isomorphism. For concision, we will hereafter let $M(Ab)$ abbreviate $M(A \cup \{b\})$, $M(Abc)$ stand for $M(A \cup \{b, c\})$, and so forth.

The fundamental way of producing matroids over R is from vector configurations in an R-module. Given a f.g. R-module N and a list $X = x_1, \ldots, x_n$ of elements of N, the matroid M_X of X associates to the sublist A of X the quotient

$$M_X(A) = N \Big/ \left(\sum_{x \in A} Rx \right). \tag{2.1}$$

For each $x \in X$ there is a quotient map from $M_X(A)$ to $M_X(A \cup \{x\})$, which quotients out by the image of Rx in $M_X(A)$; this system of maps satisfies axioms (M1) and (M2).

The following definition captures this concisely. Let $\mathcal{B}(E)$ be the category of the Boolean poset of subsets of E, where inclusions of sets are the morphisms.

Definition 2.2 *A matroid M over R is* representable *(or* realizable*) if it is the map on objects of some functor $F : \mathcal{B}(E) \to R$*-Mod*, and axioms (M1) and (M2) are satisfied by choosing the morphisms $F(A \to Ab)$. A* representation *(or* realization*) of M is a choice of such an F.*

So M_X is a representable matroid, and X gives a representation thereof. We have chosen to cast Definition 2.2 as we did, as opposed to in a more down-to-earth way involving M_X, to emphasize the way in which a representable matroid is a matroid. A representation of a matroid over R is a functor from $\mathcal{B}(E)$, with both objects and morphisms having images. A general matroid over R is what is gotten by retaining only the objects as data, discarding the morphisms and merely requiring that they can be resupplied to look like a represented matroid over R in any square of covering relations in $\mathcal{B}(E)$.

Fact 2.3 *If a matroid M over R is representable, corresponding to the functor F, then it is the matroid M_X of a vector configuration $(N, X = \{x_a\})$, where N is $F(\emptyset)$, and x_a is a generator of $\ker F(\emptyset \to \{a\})$ for each $a \in E$. Indeed, in this above setting, the pushout axiom (M2) applied to F guarantees that equation* (2.1) *holds for all $A \subseteq E$.*

Our having chosen to call these objects "matroids over R" is appropriate, as they are a generalization of matroids in the classical sense, as we show in Proposition 2.5. There is one hitch in the equivalence, corresponding to the ability to choose a vector configuration that does not span its ambient space. Accordingly, let us say that a matroid M over R is *full-rank* if no nontrivial projective module is a direct summand of $M(E)$. Lemma 2.4 shows that very little is lost in restricting to full-rank matroids.

Before getting there we must generalize some standard operations on matroids. Let M and M' be matroids over R on respective ground sets E and E'. We define their *direct sum* $M \oplus M'$ on the ground set $E \amalg E'$ by

$$(M \oplus M')(A \amalg A') = M(A) \oplus M'(A').$$

If i is an element of E, we define two matroids over R on the ground set $E \setminus \{i\}$: the *deletion* of i in M, denoted $M \setminus i$, by

$$(M \setminus i)(A) = M(A)$$

and the *contraction* of i in M, denoted $M \setminus i$, by

$$(M/i)(A) = M(A \cup \{i\}).$$

It is easily seen that the class of representable matroids is closed under minors and direct sums.

If N is an R-module, let the *empty matroid* for N be the matroid over R on the ground set $\emptyset$ which maps $\emptyset$ to N. By a *projective empty matroid* we mean an empty matroid for a projective module.

Lemma 2.4 *Every matroid M over R is the direct sum of a full-rank matroid over R and a projective empty matroid.*

Recall that the *corank* $\operatorname{cork}(A)$ of a set A in a classical matroid is equal to $\operatorname{rk}(E) - \operatorname{rk}(A)$, where $\operatorname{rk}(E)$ is the rank of the matroid.

Proposition 2.5 *Let $\mathbb{K}$ be a field. Full-rank matroids M over $\mathbb{K}$ are equivalent to (classical) matroids. If M is a full-rank matroid over $\mathbb{K}$, then $\dim M(A)$ is the corank of A in the corresponding classical matroid. Furthermore, a matroid over $\mathbb{K}$ is representable if and only if, as a classical matroid, it is representable over $\mathbb{K}$.*

The proof of this fact is simple, and relies on the fact that finitely generated modules over $\mathbb{K}$ are the finite-dimensional $\mathbb{K}$-vector spaces, which are completely classified up to isomorphism by dimension. So we may replace $M(A)$ by its $\mathbb{K}$-dimension without losing information.

Let $R \to S$ be a map of rings. Then every matroid over S is naturally also a matroid over R. Furthermore, given such a map $R \to S$, the tensor product $— \otimes_R S$ is a functor R-Mod $\to$ S-Mod. One can use this to perform base change of matroids over R. If M is a matroid over R, define $M \otimes_R S$ be the composition of M with $— \otimes_R S$, so that for every A

$$(M \otimes_R S)(A) = M(A) \otimes_R S.$$

Proposition 2.6 *If M is a matroid over R, then $M \otimes_R S$ is a matroid over S.*

Two special cases of this construction will be of fundamental importance for our theory.

1. For every prime ideal $\mathfrak{m}$ of R, let $R_\mathfrak{m}$ be the localization of R at $\mathfrak{m}$. We call $M_\mathfrak{m} \doteq M \otimes_R R_\mathfrak{m}$ the *localization* of M at $\mathfrak{m}$.

2. If R is an integral domain, let $\mathrm{Frac}(R)$ be the fraction field of R. Then we call $M_{\text{gen}} \doteq M \otimes_R \mathrm{Frac}(R)$ the *generic matroid* of M.

Our approach will be much based on studying the matroid M via these localizations.

Notice that every matroid over $R_\mathfrak{m}$ induces a matroid over the residue field $R_\mathfrak{m}/(\mathfrak{m})$; the latter, as well as M_{gen}, is by Proposition 2.5 equivalent to a classical matroid (except that it may be not full-rank).

3 Dedekind domains

In several ways, Definition 2.1 yields a theory best parallelling the theory of classical matroids just when R is a Dedekind domain. The reason for that is explained in [10, Lemma 3.1 and Example 3.2].

We next recall some structural results about modules over a Dedekind domain R. Given an R-module N, let $N_{\mathrm{tors}} \subseteq N$ denote the submodule of its torsion elements, and N_{proj} denote the projective module N/N_{tors}. Then N is the direct sum of N_{tors} and of a projective module isomorphic to N_{proj}. We recall the following fact.

Proposition 3.1 *[9, exercises 19.4–6] Every torsion R-module may be written uniquely up to isomorphism as a sum of submodules $R/\mathfrak{m}^k$ for $\mathfrak{m}$ a maximal prime of R and $k \in \mathbb{Z}_{>0}$.*

Every nonzero projective R-module is uniquely isomorphic to $R^h \oplus I$ for some $h \geq 0$ and nonzero ideal I, up to differing isomorphic choices of I. For ideals I and J, we have $I \oplus J \cong R \oplus (I \otimes J)$.

We recall the following definitions. The *Picard group* of R, $\mathrm{Pic}(R)$, is the set of the isomorphism classes of f.g. projective modules of rank 1, with product induced by the tensor product. If P is a projective module of rank n, the exterior algebra $\Lambda^n P$ is a f.g. projective module of rank $\binom{n}{n} = 1$. We call *determinant*, and denote by $\det(P)$, its class in $\mathrm{Pic}(R)$.

We will also find useful a description of the *algebraic K-theory group* $K_0(R)$ of f.g. R-modules: that is, the abelian group generated by isomorphism classes $[N]$ of f.g. R-modules, modulo the relations $[N] = [N'] + [N'']$ for any exact sequence

$$0 \to N' \to N \to N'' \to 0.$$

Proposition 3.2 *There is an isomorphism of groups*

$$\Phi : K_0(R) \longrightarrow \mathbb{Z} \oplus \mathrm{Pic}(R).$$

In fact, when P is a projective module, the map above is simply given by $\Phi([P]) = (\mathrm{rk}(P), \det(P))$.

In virtue of the isomorphism above, from now on we will denote by $\det(N)$ the class of any f.g. R-module N in the Picard group, i.e. the second summand of $\Phi([N])$. In the same way, by $\mathrm{rk}(N)$ we denote the first summand of $\Phi([N])$: this coincides with the rank of N_{proj}, i.e. with the dimension of $N \otimes \mathrm{Frac}(R)$.

Note in particular that Φ extends the usual map from invertible ideals to $\mathrm{Pic}(R)$.

The potential nontriviality of this summand $\mathrm{Pic}(R) \subseteq K_0(R)$ has global consequences for matroids over R: see Proposition 4.3 below.

4 Duality for matroids over Dedekind domains

In this section R will be a Dedekind domain. Let M be a matroid over R, on ground set E. Fix a free module F that surjects on $M(\emptyset)$. For any $A \subseteq E$ and maximal flag of subsets $\emptyset = A_0 \subsetneq A_1 \subsetneq \cdots \subsetneq A_{|A|} = A$, we obtain a composite surjection

$$F \to M(\emptyset) \to M(A_1) \to \cdots \to M(A).$$

Using the horseshoe lemma, we may assemble minimal projective resolutions of each step of this composition into a projective resolution of $F/M(A)$, yielding a projective resolution

$$P(A)_\bullet : \quad 0 \to P_2(A) \to P_1(A) \xrightarrow{d_1} F \to M(A) \to 0$$

of $M(A)$. As usual, we write $^\vee$ for the contravariant functor $\mathrm{Hom}(—, R)$.

Definition 4.1 *Define the module $M^*(E \setminus A)$ as the cokernel of the map dual to d_1 in $P(A)_\bullet$, that is*

$$M^*(E \setminus A) \doteq \mathrm{coker}\left(F^\vee \xrightarrow{d_1^\vee} P_1(A)^\vee\right).$$

This is well-defined ([10, Lemma 4.2]). We define M^*, the dual *matroid over R to M, to be the collection of these modules $M^*(E \setminus A)$.*

Theorem 4.2 *If R is a Dedekind domain, and M is a matroid over R, then its dual M^* is a full-rank matroid over R. Furthermore, M is the direct sum of M^{**} and the projective empty matroid for $M(E)_{\mathrm{proj}}$; in particular, if M is full-rank, $M^{**} = M$.*

If M is representable, also M^* is.

The last statement above gives a generalization of the classical Gale duality of vector configurations.

Furthermore, duality of matroids over rings is well-behaved with respect to deletion, contraction, direct sums, and tensor products, as shown in [10, Proposition 4.9].

Proposition 4.3 *Let M be a matroid over R. The element*

$$\det(M) \doteq \det(M(A)_{\mathrm{proj}}) + \det(M^*(E \setminus A)_{\mathrm{proj}}) + \det(M(A)_{\mathrm{tors}})$$

of $\mathrm{Pic}(R)$ is independent of the choice of $A \subseteq E$.

5 Structure of matroids over a DVR

In this section and the next we record some structure theorems for matroids over R in terms of structure theorems for the modules over R themselves. Our analysis of general Dedekind domains in the next section will make much use of base changing to localizations of R, so we begin here with the local case.

For the whole of this section, R will be a DVR with maximal ideal $\mathfrak{m}$. We first recall the structure theory of f.g. R-modules: any indecomposible f.g. R-module is isomorphic to either R or $R/\mathfrak{m}^n$ for some integer $n \geq 1$. We will sometimes formally subsume R into the latter family by writing it as $R/\mathfrak{m}^\infty$. So, if N is a f.g. R-module and $i \geq 1$ is an integer, define

$$d_i(N) \doteq \dim_{R/\mathfrak{m}}(\mathfrak{m}^{i-1}N/\mathfrak{m}^i N),$$

and $d_{\leq i}(N) \doteq \sum_{j=1}^{i} d_j(N)$, and for convenience $d_i(N) = d_{\leq i}(N) = 0$ if $i \leq 0$. Let $d_\bullet(N)$ denote the infinite sequence of these. We have

$$d_i(R/\mathfrak{m}^n) = \begin{cases} 1 & 0 < i \leq n \\ 0 & i > n \end{cases},$$

where n may be ∞. The following is a quick consequence.

Proposition 5.1 *Isomorphism types of f.g. R-modules are in bijection with nonincreasing infinite sequences $d_\bullet$ of nonnegative integers indexed by positive integers, the bijection being given by*

$$N \longleftrightarrow d_\bullet(N).$$

This bijection permits a straightforward identification of those isomorphism classes of modules which permit maps satisfying axioms (M1) and (M2).

Theorem 5.2 *Let N and N' be f.g. R-modules. There exists a surjection $\phi : N \to N'$ with cyclic kernel if and only if*

(L1) for each $n \geq 1$,

$$d_n(N) - d_n(N') \in \{0, 1\}.$$

Let $M(\emptyset)$, $M(1)$, $M(2)$, and $M(12)$ be f.g. R-modules. There exist four surjections with cyclic kernels forming a pushout square

$$\begin{tikzcd} M(\emptyset) \arrow[r, "\phi"] \arrow[d, "\psi"'] \arrow[dr, phantom, "\lrcorner"] & M(1) \arrow[d, "\psi'"] \\ M(2) \arrow[r, "\phi'"'] & M(12) \end{tikzcd}$$

if and only if (L1) is satisfied for each pair $M(A), M(Ab)$, and moreover

(L2a) for each $n \geq 1$,

$$d_{\leq n}(M(\emptyset)) - d_{\leq n}(M(1)) - d_{\leq n}(M(2)) + d_{\leq n}(M(12)) \geq 0;$$

(L2b) for any $n \geq 1$ such that $d_n(M(1)) \neq d_n(M(2))$, equality holds above:

$$d_{\leq n}(M(\emptyset)) - d_{\leq n}(M(1)) - d_{\leq n}(M(2)) + d_{\leq n}(M(12)) = 0.$$

Condition (L2a) asserts that $A \mapsto -d_{\leq n}(M(A))$ is a *submodular* function.

In the case that N and N' have finite length, condition (L1) follows from facts about the Hall algebra [11]. Indeed, it is equivalent that N have finite length and that $d_i(N)$ stabilize to 0 for $i \gg 0$. In this case d_i is a partition, and its conjugate partition is the one usually used to label N. For a cyclic module, this conjugate partition has a single row. Then, under the specialization taking the Hall polynomials to the Littlewood-Richardson coefficients, condition (L1) is a consequence of the Pieri rule.

The structure of matroids over R in fact has interesting tropical-geometric import (for background on tropical geometry, see [12]). The first inkling of this is in three-element matroids:

Proposition 5.3 *Let M be a matroid over R on the ground set $[3]$, and let n be a natural or ∞. Then, among the three quantities*

$$d_{\leq n}(M(1)) + d_{\leq n}(M(23)), d_{\leq n}(M(2)) + d_{\leq n}(M(13)), d_{\leq n}(M(3)) + d_{\leq n}(M(12)),$$

the minimum is achieved at least twice.

Let M be a matroid over R with ground set E. For $A \subseteq E$, define p_A to be $d_{\leq n}(M(A))$. Proposition 5.3 applied to the 3-element minors of M can be taken to say that the *tropicalizations* of the relations

$$p_{Ab}p_{Acd} - p_{Ac}p_{Abd} + p_{Ad}p_{Abc} = 0 \tag{5.1}$$

hold of the numbers $p_\bullet$, where we continue abbreviating $A \cup \{b, c\}$ as Abc and similarly.

The relations (5.1) are among the Plücker relations for the full flag variety (of type A). We can show [10, Proposition 5.6] that the $p_\bullet$ satisfy some of the other Plücker relations, which imply that for every r the point $(p_A : |A| = r)$ lies on the *Dressian* $\mathrm{Dr}(r, n)$. The Dressian is one Grassmannian-like space in tropical geometry: it is the parameter space for tropical linear spaces. That is, there is a tropical linear space determined by $(p_A : |A| = r)$. Corollary 5.4 follows.

Corollary 5.4 *Let M be a matroid over a DVR $(R, \mathfrak{m})$. Then the function $A \mapsto \dim_{R/\mathfrak{m}} M(A)$ makes the generic matroid of M into a valuated matroid, in the sense of Dress and Wenzel [8].*

To be precise, our sign convention is the opposite of the one adopted in [8]; for perfect agreement we would have to negate this function. But our sign convention is frequently adopted in tropical geometry.

Conjecture 5.5 *The collection of the p_A satisfies every tropical Plücker relation, i.e. gives a point on the Dressian analogue of the tropical full flag variety.*

We expect that Conjecture 5.5 follows directly from Proposition 5.3, and needs no further matroidal arguments. The main obstruction to proving 5.5 seems to be only that the tropical full flag variety has been little studied.

6 Global structure of matroids over a Dedekind domain

Throughout this section R will be a Dedekind domain. Let us recall that given a R-module N, by $\det(N)$ we will denote its class in the Picard group $\mathrm{Pic}(R)$, as defined in Section 3. The next Theorem gives a complete characterization of the structure of matroids over R, in terms of their localizations for which we have Theorem 5.2.

Theorem 6.1 *Let N and N' be f.g. R-modules. There exists a surjection $N \to N'$ with cyclic kernel if and only if there exists such a surjection $N_{\mathfrak{m}} \to N'_{\mathfrak{m}}$ after localizing at each maximal prime $\mathfrak{m}$ of R, and*

- *if* $\mathrm{rk}(N) - \mathrm{rk}(N') = 0$ *then* $\det(N_{\mathrm{proj}}) = \det(N'_{\mathrm{proj}})$, *whereas*
- *if* $\mathrm{rk}(N) - \mathrm{rk}(N') = 1$ *then* $\det(N) = \det(N')$.

Let $M(\emptyset)$, $M(1)$, $M(2)$, and $M(12)$ be f.g. R-modules. There exist four surjections with cyclic kernels forming a pushout square

$$\begin{tikzcd} M(\emptyset) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & M(1) \arrow[d] \\ M(2) \arrow[r] & M(12) \end{tikzcd}$$

if and only if the same is true after localizing at each maximal prime $\mathfrak{m}$, and the above conditions on classes are true of each $(N, N') = (M(A), M(Ab))$.

6.1 Quasi-arithmetic matroids

If M is a matroid over $\mathbb{Z}$, then we can define a corank function of M as the corank function of the generic matroid $M \otimes_{\mathbb{Z}} \mathbb{Q}$ described above, that is $\mathrm{cork}(A) = \mathrm{rk}_{\mathbb{Z}}(M(A)_{\mathrm{proj}})$.

As before, we let $M(A)_{\mathrm{tors}}$ denote the torsion submodule (subgroup, in this case) of $M(A)$. Then we define

$$m(A) \doteq |M(A)_{\mathrm{tors}}|.$$

Corollary 6.2 *The triple (E, cork, m) is a* quasi-arithmetic matroid*, i.e (E, cork) defines a matroid, and m satisfies the following properties:*

(A1) *Let be $A \subseteq E$ and $b \in E$; if b is dependent on A, then $m(A \cup \{b\})$ divides $m(A)$; otherwise $m(A)$ divides $m(A \cup \{b\})$;*

(A2b) *if $A \subseteq B \subseteq E$ and B is a disjoint union $B = A \cup F \cup T$ such that for all $A \subseteq C \subseteq B$ we have $\mathrm{rk}(C) = \mathrm{rk}(A) + |C \cap F|$, then*

$$m(A) \cdot m(B) = m(A \cup F) \cdot m(A \cup T).$$

Furthermore it satisfies the following property:

(A2a) *if $A, B \subseteq E$ and $\mathrm{rk}(A \cup B) + \mathrm{rk}(A \cap B) = \mathrm{rk}(A) + \mathrm{rk}(B)$, then $m(A) \cdot m(B)$ divides $m(A \cup B) \cdot m(A \cap B)$*

In fact properties (A1), (A2a), (A2b) follow from (L1), (L2a), (L2b) respectively. This corollary establishes that matroids over $\mathbb{Z}$ recover many of the essential features of the second author's theory of *arithmetic matroids* from [5]. To be precise, the objects we have recaptured are *quasi-arithmetic matroids*: see [10, Remark 6.4]. In fact the two objects are not truly equivalent, in that the information contained in matroids over $\mathbb{Z}$ is richer, because there are many finite abelian groups with the same cardinality.

7 The Tutte-Grothendieck group

In this section we continue to let R be a Dedekind domain. All matroids over R in this section are full-rank. As we defined the operations of deletion and contraction in Section 2, any element may be deleted or contracted. However, suppose $a \in E$ is a (*generic*) *coloop* of a matroid M over R, that is a coloop of the generic matroid, equivalently an element such that $M(E \setminus \{a\})$ has a nontrivial projective summand. In this case, $M \setminus a$ is not full-rank. The dual of this situation is the case where a is a (*generic*) *loop*, i.e. a loop of the generic matroid, and one contracts a.

Essentially following Brylawski [4], define the *Tutte-Grothendieck group* of matroids over R, which we here denote $K(R\text{-Mat})$, to be the abelian group generated by a symbol $\mathbf{T}_M$ for each unlabelled full-rank matroid M over R with nonempty ground set, modulo the relations

$$\mathbf{T}_M = \mathbf{T}_{M\setminus a} + \mathbf{T}_{M/a}$$

whenever a is not a generic loop or coloop (so that we avoid the above situations). We have omitted empty matroids for technical reasons, though they cause no essential problem; the interested reader can refer to [10, Remark 7.2]. By "unlabelled", we mean that we consider two matroids M and M' over R to be identical if there is a bijection $\sigma : E \xrightarrow{\sim} E'$ of their ground sets such that $M(A) \cong M'(\sigma(A))$ for each subset A of E.

The ring $K(R\text{-Mat})$ turns out to be best understood in terms of the monoid ring of the monoid of R-modules under direct sum, as in Theorem 7.1 below. Define $\mathbb{Z}[R\text{-Mod}]$ to be the ring with a $\mathbb{Z}$-linear basis $\{u^N\}$ with an element u^N for each f.g. R-module N up to isomorphism, and product given by $u^N u^{N'} = u^{N\oplus N'}$.

Theorem 7.1 *The Tutte-Grothendieck group $K(R\text{-Mat})$ is a ring without unity, with product given by $\mathbf{T}_M \cdot \mathbf{T}_{M'} = \mathbf{T}_{M\oplus M'}$. As a ring it injects into $\mathbb{Z}[R\text{-Mod}] \otimes \mathbb{Z}[R\text{-Mod}]$, and under this injection, the class of M maps to*

$$\mathbf{T}_M = \sum_{A\subseteq E} X^{M(A)} Y^{M^*(E\setminus A)}, \tag{7.1}$$

where $\{X^N\}$ and $\{Y^N\}$ are the respective bases of the two tensor factors $\mathbb{Z}[R\text{-Mod}]$.

If we include empty matroids, the ring $\mathbb{K}(R\text{-Mat})$ is the subring of $\mathbb{Z}[R\text{-Mod}] \otimes \mathbb{Z}[R\text{-Mod}]$ generated by X^P and Y^P as P ranges over rank 1 projective modules, and $(XY)^N$ as N ranges over torsion modules.

Here $(XY)^N$ abbreviates $X^N Y^N$. We immediately compare Theorem 7.1 with the case of matroids over a field, where the Tutte-Grothendieck invariant is the familiar Tutte polynomial $\mathbf{T}_M$. If R is a field, then $\mathbb{Z}[R\text{-Mod}]$ is the univariate polynomial ring $\mathbb{Z}[u]$, and then $\mathbb{Z}[R\text{-Mod}] \otimes \mathbb{Z}[R\text{-Mod}]$ is, appropriately, a bivariate polynomial ring. If we call the generators of the two tensor factors $x - 1$ and $y - 1$ rather than X and Y, then equation (7.1) in fact gives the classical Tutte polynomial, since $\dim M(A)$ is the corank of A and $\dim M^*(E \setminus A)$ is its nullity.

Since decomposing a matroid M over a ring into $M \setminus i$ and M/i is not a unique decomposition in the sense of [4], and the irreducibles for direct sum are not all single-element matroids, Theorem 7.1 does not follow directly from the bidecomposition methods of [4].

7.1 Arithmetic Tutte polynomial and quasi-polynomial

In this subsection, M is a matroid over $\mathbb{Z}$. We show that the arithmetic Tutte polynomial of its associated quasi-arithmetic matroid $\hat{M}$, and its Tutte quasi-polynomial, are each images of $\mathbf{T}_M$ under ring homomorphisms. When $R = \mathbb{Z}$, the Picard group is trivial, and

$$\mathbf{T}_M = \sum_{A \subseteq E} (X^R)^{\mathrm{cork}_M(A)} (Y^R)^{\mathrm{nullity}_M(A)} (XY)^{M(A)_{\mathrm{tors}}}.$$

where we use the notation $\mathrm{nullity}_M(A) = \mathrm{cork}_{M^*}(E \setminus A) = \dim M^*(E \setminus A)$.

We may define a specialization of $\mathbf{T}_M$ by specializing X^R to $(x-1)$, Y^R to $(y-1)$, and $(XY)^N$ to the cardinality of N for each torsion module N. This specialization is the arithmetic Tutte polynomial $\mathbf{M}_{\hat{M}}(x, y)$ of the quasi-arithmetic matroid $\hat{M}$ defined by M:

$$\mathbf{M}_{\hat{M}}(x, y) = \sum_{A \subseteq E} m(A)(x-1)^{\mathrm{rk}(E)-\mathrm{rk}(A)}(y-1)^{|A|-\mathrm{rk}(A)}.$$

This polynomial proved to have several applications to toric arrangements, partition functions, zonotopes, and graphs with labeled edges (see [13, 6, 5]). Notice that an ordinary matroid $\tilde{M}$ can be trivially made into an arithmetic matroid $\hat{M}$ by setting all the multiplicities to be equal to 1, and then $\mathbf{M}_{\hat{M}}(x, y)$ is nothing but the classical Tutte polynomial $\mathbf{T}_{\tilde{M}}(x, y)$.

The polynomial $\mathbf{M}_{\hat{M}}(x, y)$ is not the universal deletion-contraction invariant of $\hat{M}$: for instance, the ordinary Tutte polynomial $\mathbf{T}_{\tilde{M}}(x, y)$ of the matroid $\tilde{M}$ obtained from $\hat{M}$ by forgetting of its arithmetic data is also a deletion-contraction invariant of $\hat{M}$, which is not determined by $\mathbf{M}_{\hat{M}}(x, y)$. This led the authors of [3] to define a *Tutte quasi-polynomial* $\mathbf{Q}_M(x, y)$, interpolating between $\mathbf{T}_{\tilde{M}}(x, y)$ and $\mathbf{M}_{\hat{M}}(x, y)$. This invariant is stronger, but still not universal, and more importantly, it is not an invariant of the arithmetic matroid, as it depends on the groups $M(A)_{\mathrm{tors}}$ and not just on their cardinalities. In fact $\mathbf{Q}_M(x, y)$ is an invariant of the matroid over $\mathbb{Z}$, and we show explicitly how to compute it from the universal invariant.

For every positive integer q, let us define a function V_q as $V_q((XY)^{\mathbb{Z}/p^k}) = 1$ if p^k divides q, while $V_q((XY)^{\mathbb{Z}/p^k}) = p^{k-j}$ if p^k does not divide q and $j \geq 0$ is the largest integer such that p^j divides q. We will extend this to define $V_q((XY)^N)$ multiplicatively for any torsion abelian group N. Then we define a specialization of $\mathbf{T}_M$ to the ring of quasipolynomials by specializing X^R to $(x-1)$, Y^R to $(y-1)$, and $(XY)^N$ to $V_{(x-1)(y-1)}((XY)^N)$. This gives

$$\mathbf{Q}_M(x, y) = \sum_{A \subseteq E} \frac{|M(A)_{\mathrm{tors}}|}{|(x-1)(y-1)M(A)_{\mathrm{tors}}|}(x-1)^{\mathrm{rk}(E)-\mathrm{rk}(A)}(y-1)^{|A|-\mathrm{rk}(A)}.$$

Since $(q + |G|)G = qG$ holds for any finite group G, it follows that $\mathbf{Q}_M(x, y)$ is a quasi-polynomial in $q = (x-1)(y-1)$. In particular, when $|M(A)_{\mathrm{tors}}|$ divides $(x-1)(y-1)$, then the group $(x-1)(y-1)M(A)_{\mathrm{tors}}$ is trivial and $\mathbf{Q}_M(x, y)$ coincides with $\mathbf{M}_{\hat{M}}(x, y)$; while when $|M(A)_{\mathrm{tors}}|$ is coprime with

$(x-1)(y-1)$, then $\mathbf{Q}_M(x,y)$ coincides with $\mathbf{T}_{\tilde{M}}(x,y)$. Then in some sense $\mathbf{Q}_M(x,y)$ interpolates between the two polynomials.

Notice that while $\mathbf{M}_{\hat{M}}$ and $\mathbf{T}_{\tilde{M}}(x,y)$ only depend on the induced quasi-arithmetic matroid $\hat{M}$, $\mathbf{T}_M$ and $\mathbf{Q}_M(x,y)$ are indeed invariants of the matroid over $\mathbb{Z}$, M. Also the *chromatic quasi-polynomial* and the *flow quasi-polynomial* defined in [3] are actually invariants of the matroid over $\mathbb{Z}$: by [3, Theorem 9.1] they are specializations of $\mathbf{Q}_M(x,y)$, and hence of the universal invariant $\mathbf{T}_M$.

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