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Moments of Askey-Wilson polynomials

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Abstract. New formulas for the n^{th} moment $\mu_n(a,b,c,d;q)$ of the Askey-Wilson polynomials are given. These are derived using analytic techniques, and by considering three combinatorial models for the moments: Motzkin paths, matchings, and staircase tableaux. A related positivity theorem is given and another one is conjectured.

Résumé. Nous présentons de nouvelles formules pour les n-moments $\mu_n(a,b,c,d;q)$ des polynômes Askey-Wilson. Ils sont calculés avec des techniques analytiques, et en considérant trois modèles combinatoires pour les moments: des chemins de Motzkin, des couplages, et des tableaux escalier. Un théorème de positivité liée est donné et un autre est conjecturé.

Keywords: Askey-Wilson polynomials, moments of orthogonal polynomials, Motzkin paths, hypergeometric series

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1 Introduction

The monic Askey-Wilson polynomials $P_n = P_n(x;a,b,c,d;q)$ are polynomials in x of degree n which depend upon five parameters a, b, c, d, and q. They may be defined by the three-term recurrence $P_{n+1} = (x-b_n)P_n - \lambda_n P_{n-1}$ with $P_{-1}=0$ and $P_0=1$ for $b_n = \frac{1}{2}(a+a^{-1}-(A_n+C_n))$ and $\lambda_n = \frac{1}{4}A_{n-1}C_n$, where

$$A_n = \frac{(1-abq^n)(1-acq^n)(1-adq^n)(1-abcdq^{n-1})}{a(1-abcdq^{2n-1})(1-abcdq^{2n})},$$
$$C_n = \frac{a(1-q^n)(1-bcq^{n-1})(1-bdq^{n-1})(1-cdq^{n-1})}{(1-abcdq^{2n-2})(1-abcdq^{2n-1})}.$$

We refer to [6] for the standard basic hypergeometric notation and for information about the Askey-Wilson polynomials.

In view of the above three-term recurrence relation, the Askey-Wilson polynomials are orthogonal polynomials. An explicit absolutely continuous measure [1, Theorem 2.2], may be given for these polynomials. (We assume here that $\max\{|a|,|b|,|c|,|d|\} < 1$.) It is supported on $x\in[-1,1]$, and with $x=\cos\theta$, the measure is

$$w(\cos\theta,a,b,c,d;q) = \frac{(e^{2i\theta})_\infty(e^{-2i\theta})_\infty}{(ae^{i\theta})_\infty(ae^{-i\theta})_\infty(be^{i\theta})_\infty(be^{-i\theta})_\infty(ce^{i\theta})_\infty(ce^{-i\theta})_\infty(de^{i\theta})_\infty(de^{-i\theta})_\infty}.$$

The measure has total mass given by the *Askey-Wilson integral*,

$$I_0 = \frac{(q)_\infty}{2\pi}\int_0^\pi w(\cos\theta, a, b, c, d; q)d\theta = \frac{(abcd)_\infty}{(ab)_\infty(ac)_\infty(ad)_\infty(bc)_\infty(bd)_\infty(cd)_\infty}. \tag{1}$$

The purpose of this paper is to study the n^{th} moment $\mu_n(a,b,c,d;q)$ of the measure $w(x;a,b,c,d;q)$ for the Askey-Wilson polynomials

$$\mu_n(a,b,c,d;q) = C\int_{-1}^1 x^n w(x;a,b,c,d;q)\frac{dx}{\sqrt{1-x^2}}.$$

for some normalization constant C. With the normalization of $\mu_0(a,b,c,d;q)=1$, (the explicit C may be found from (1)), the n^{th} moment is known to be a rational function of a, b, c, d and q. We shall give new explicit expressions for $\mu_n(a,b,c,d;q)$ and study three combinatorial models for $\mu_n(a,b,c,d;q)$. One unusual feature of these results is the mixture of binomial and q-binomial terms in the explicit formulas. We shall see why this occurs, both analytically and combinatorially.

The simplest expression for $\mu_n(a,b,c,d;q)$ is a double sum (see [4, Theorem 1.12])

$$\mu_n(a,b,c,d;q) = \frac{1}{2^n}\sum_{m=0}^n \frac{(ab,ac,ad;q)_m}{(abcd;q)_m}q^m \sum_{j=0}^m \frac{q^{-j^2}a^{-2j}(aq^j+q^{-j}/a)^n}{(q,q^{1-2j}/a^2;q)_j(q,q^{2j+1}a^2;q)_{m-j}}. \tag{2}$$

However this expression is not obviously symmetric in a, b, c, and d, even though the Askey-Wilson polynomials $P_n(x;a,b,c,d;q)$ and the moments $\mu_n(a,b,c,d;q)$ are symmetric. Nor does it exhibit the correct poles of $\mu_n(a,b,c,d;q)$ as a rational function. For $d=0$ the moments are polynomials in a, b, c, and q. We give new expressions for the moments $\mu_n(a,b,c,d;q)$, which are symmetric and polynomial when $d=0$, see Theorem 2.3. We also give a symmetric version for all a,b,c,d in Theorem 2.7, although the polynomial dependence in q is not clear. We give new expressions for the moments $\mu_n(a,b,c,d;q)$ in the special case $b=-a$, $d=-c$, see Theorem 2.4 and Theorem 2.5. We prove a new positivity theorem in Corollary 4.4, and conjecture another one in Conjecture 1.

Our second goal is to combinatorially study the moments $\mu_n(a,b,c,d;q)$ as functions of a, b, c, and d. We use three combinatorial models for this purpose. The moments for any set of orthogonal polynomials may be given as weighted Motzkin paths [13]. In this case $\mu_n(a,b,c,d;q)$ is the generating function for Motzkin paths with weights which are rational functions of a,b,c,d and q. We use this setup and a generalization of an idea of D. Kim [11] to combinatorially prove Theorem 3.1 and Corollary 3.2 in Section 3. A special combinatorial model for the Askey-Wilson integral, which evaluated the normalization constant C, was given in [7]. We modify this model appropriately to give a combinatorial model for some non-normalized moments in Theorem 4.1. We also give new explicit rational expressions for $\mu_n(a,b,c,d;q)$ so that $(abcd)_n\mu_n(a,b,c,d;q)$ are clearly polynomials in a, b, c, d and q, see Theorem 4.2 and Theorem 4.3.

A third combinatorial model was given by Corteel and Williams [5]. They give a combinatorial interpretation for the polynomial $2^n(abcd)_n\mu_n(a,b,c,d;q)$ using a rational transformation over the complex numbers of the parameters a, b, c, and d to parameters α, β, γ, and δ. In these new parameters $2^n(abcd)_n\mu_n(a,b,c,d;q)$ is a polynomial with positive integer coefficients with a combinatorial meaning. (See Section 5). Using their ideas, we explicitly find coefficients of certain terms in $2^n(abcd)_n\mu_n(a,b,c,d;q)$ as Catalan numbers in Theorem 5.2.

2 Askey-Wilson moments

In this section we consider the moments $\mu_n(a,b,c,d;q)$ as functions of the parameters a, b, c, d and q. Our goal is to give new explicit formulas for these moments, using simple series and integral evaluations. First we note an elementary fact.

Proposition 2.1. *$2^n(abcd;q)_n\mu_n(a,b,c,d;q)$ is a polynomial in a,b,c,d,q with integer coefficients.*

Note that the expression (2) has some removable singularities as functions of a and q. The method of proof of (2), which appears in [4], is to use an appropriate q-Taylor expansion. Theorem 2.2 and Theorem 2.3 below follow from (2) by series manipulations which are not given here.

Using (2) we have the following theorem, which exhibits $2^n(abcd;q)_n\mu_n(a,b,c,d;q)$ as a symmetric polynomial in b, c, and d.

Theorem 2.2. *The Askey-Wilson moments are*

$$2^n\mu_n(a,b,c,d;q)=\sum_{m=0}^{n}\frac{(ab,ac,ad;q)_m}{(abcd;q)_m}(-q)^m\sum_{s=0}^{n+1}\left(\binom{n}{s}-\binom{n}{s-1}\right)\\ \times\sum_{p=0}^{n-2s-m}a^{-n+2s+2p}\begin{bmatrix}m+p\\ m\end{bmatrix}_q\begin{bmatrix}n-2s-p\\ m\end{bmatrix}_q q^{(-n+2s+p)m+\binom{m}{2}}.$$

If k is not a nonnegative integer, we define $\binom{n}{k}=0$.

Theorem 2.3. *The Askey-Wilson moments for $d=0$ are*

$$2^n\mu_n(a,b,c,0;q)=\sum_{k=0}^{n}\left(\binom{n}{\frac{n-k}{2}}-\binom{n}{\frac{n-k}{2}-1}\right)\\ \times\sum_{u+v+w+2t=k}a^ub^vc^w(-1)^tq^{\binom{t+1}{2}}\begin{bmatrix}u+v+t\\ v\end{bmatrix}_q\begin{bmatrix}v+w+t\\ w\end{bmatrix}_q\begin{bmatrix}w+u+t\\ u\end{bmatrix}_q,$$

where the second sum is over all integers $0\le u,v,w\le k$ and $-k\le t\le k/2$ satisfying $u+v+w+2t=k$.

A special case of the Askey-Wilson polynomials has a different expression for the moments. Consider $b=-a$ and $d=-c$, so that the Askey-Wilson measure is an even function. In this case $b_n=0$, so the odd moments are zero, and the $2n^{\text{th}}$ moment has a shorter alternative expression, again proven by q-Taylor series and integration.

Theorem 2.4. *The non-zero Askey-Wilson moments for $b=-a$ and $d=-c$ are*

$$4^n\mu_{2n}(a,-a,c,-c;q)=\sum_{m=0}^{n}\frac{(-a^2;q)_{2m}(a^2c^2;q^2)_m}{(qa^2c^2;q^2)_m}q^{2m}\sum_{j=0}^{m}\frac{a^{-4j}q^{-2j^2}(aq^j+a^{-1}q^{-j})^{2n}}{(q^2,a^4q^{2+4j};q^2)_{m-j}(q^2,a^{-4}q^{2-4j};q^2)_j}.$$

Zeng [14] found a formula equivalent to Theorem 2.4 when $c=0$. Theorem 2.4 has a version with differences of binomial coefficients, similar to Theorem 2.2.

Theorem 2.5. *The non-zero Askey-Wilson moments for $b=-a$ and $d=-c$ are*

$$4^n\mu_{2n}(a,-a,c,-c;q)=\sum_{m=0}^{n}\frac{(-a^2;q)_{2m}(a^2c^2;q^2)_m}{(qa^2c^2;q^2)_m}(-q^2)^m$$
$$\times\sum_{s=0}^{2n+2}\left(\binom{2n+1}{s}-\binom{2n+1}{s-1}\right)\sum_{p=0}^{n-m-s}a^{-2n+4p+2s}\begin{bmatrix}m+p\\ m\end{bmatrix}_{q^2}\begin{bmatrix}n-p-s\\ m\end{bmatrix}_{q^2}q^{-2m(n-p-s)+m(m-1)}.$$

There is a positivity conjecture for the moments $\mu_{2n}(a,-a,c,-c;q)$. These moments are not polynomials, but Proposition 2.1 implies that $4^n(a^2c^2;q)_{2n}\mu_{2n}(a,-a,c,-c;q)$ is a polynomial. Half of the apparent poles of $\mu_{2n}(a,-a,c,-c;q)$ do not occur.

Proposition 2.6. *The Askey-Wilson moments*

$$\tau_{2n}(a^2,c^2)=4^n(qa^2c^2;q^2)_n\mu_{2n}(a,-a,c,-c;q)/(1-q)^n$$

are polynomials in a^2, c^2 and q with integer coefficients. Moreover the sum of the coefficients in $\tau_{2n}(a^2,c^2)$ is $2^{2n}(2n-1)(2n-3)\cdots 1$.

Conjecture 1. The coefficients of $\tau_{2n}(a^2,c^2)$ are non-negative integers.

If $q=0$, one may show that $\tau_{2n}(a^2,c^2)$ is a non-negative polynomial by a combinatorial method. The sum of the coefficients is 2^{2n}. It is a generating function for certain non-crossing complete matchings.

Although simple, (2) does not clearly demonstrate the symmetry or polynomiality of $\mu_n(a,b,c,d;q)$ in all four parameters a, b, c and d. We next give such a formula, which generalizes Theorem 2.2.

Let A be an arbitrary parameter. Let

$${}_8W_7(m)={}_8W_7(A^2/q;A/a,A/b,A/c,A/d,q^{-m};q;abcdq^m).$$

From the definition of the ${}_8W_7$ one may show that $(aA,bA,cA,dA;q)_m\,{}_8W_7(m)$ is a symmetric polynomial in a,b,c and d.

Using Watson's transformation [6, (III.17)] of an ${}_8W_7$ to a ${}_4\phi_3$, the following apparent rational function of A and q is in fact a polynomial in each of the parameters: a, b, c, d, A, and q.

$$\frac{(aA,bA,cA,dA;q)_m}{(A^2;q)_m}{}_8W_7(m)=\sum_{j=0}^{m}\begin{bmatrix}m\\ j\end{bmatrix}_q(cd)^j(A/c,A/d;q)_j(ab;q)_j(Aaq^j,Abq^j,cd;q)_{m-j}.$$

The next result, proven using q-Taylor series and integration, gives a symmetric polynomial version for $2^n(abcd;q)_n\mu_n(a,b,c,d;q)$. Theorem 2.7 is independent of A.

Theorem 2.7.

$$2^n\mu_n(a,b,c,d;q)=\sum_{m=0}^{n}\frac{(aA,bA,cA,dA;q)_m}{(A^2,abcd;q)_m}(-q)^m{}_8W_7(m)\sum_{s=0}^{n+1}\left(\binom{n}{s}-\binom{n}{s-1}\right)$$
$$\times\sum_{p=0}^{n-2s-m}A^{-n+2s+2p}\begin{bmatrix}m+p\\ m\end{bmatrix}_q\begin{bmatrix}n-2s-p\\ m\end{bmatrix}_q q^{m(-n+2s+p)+\binom{m}{2}}.$$

If $A=a$, then Theorem 2.7 becomes Theorem 2.2. Theorem 2.7 has one defect: not all of the powers of q are positive due to the $q^{m(-n+2s+p)}$ term. The individual terms are Laurent polynomials in q.

If $A^2=q$, the p-sum in Theorem 2.7 is evaluable by the q-Vandermonde sum [6, II.6].

3 Weighted Motzkin paths

The first combinatorial model uses weighted Motzkin paths to find the moments. The weights are given by the complicated rational functions in the three-term recurrence relation. However if $c = d = 0$, these rational weights become simple polynomial weights, and the combinatorial model provided by Motzkin paths simplifies, as does the formula for the moments.

Theorem 3.1. *The Askey-Wilson moments for $c = d = 0$ are*

$$2^n \mu_n(a, b, 0, 0; q) = \sum_{k=0}^{n} \left(\binom{n}{\frac{n-k}{2}} - \binom{n}{\frac{n-k}{2} - 1} \right) \sum_{u+v+2t=k} a^u b^v (-1)^t q^{\binom{t+1}{2}} \begin{bmatrix} u+v+t \\ u, v, t \end{bmatrix}_q, \tag{3}$$

where the second sum is over all nonnegative integers u, v, t satisfying $u + v + 2t = k$.

Theorem 3.1 is equivalent to a result of Josuat-Vergès [9, Theorem 6.1.1], and has an attractive special case.

Corollary 3.2. *We have*

$$2^n \mu_n(a, q/a, 0, 0; q) = \sum_{k=0}^{n} \left(\binom{n}{\frac{n-k}{2}} - \binom{n}{\frac{n-k}{2} - 1} \right) (q/a)^k \sum_{i=0}^{k} a^{2i} q^{i(k-i-1)}.$$

In this section we sketch combinatorial proofs of Theorem 3.1 and Corollary 3.2.

The main idea is as follows. We first interpret the moment $2^n \mu_n(a, b, 0, 0; q)$ as a weighted sum of Motzkin paths. Then using Penaud's decomposition [12] we can decompose a weighted Motzkin path into a pair of paths: a Dyck prefix and another weighted Moztkin path. We map the new weighted Moztkin path to a new object: doubly striped skew shapes. These objects are a generalization of striped skew shapes introduced by D. Kim [11] in order to prove the moment formula for Al-Salam-Carlitz polynomials. We then find a sign-reversing involution on doubly striped skew shapes and show that the fixed points have a weighted sum equal to a q-trinomial coefficient. This completes the sketch of the proof of Theorem 3.1. For Corollary 3.2, we find a further cancellation on the doubly striped skew shapes which leaves only one fixed point for given size.

A *Motzkin path* is a lattice path in $\mathbb{N} \times \mathbb{N}$ from $(0, 0)$ to $(n, 0)$ consisting of up steps $(1, 1)$, down steps $(1, -1)$, and horizontal steps $(1, 0)$. We say that the *level* of a step is i if it is an up step or a down step between the lines $y = i - 1$ and $y = i$, or it is a horizontal step on the line $y = i$. A *weighted Motzkin path* is a Motzkin path in which each step has a certain weight. The *weight* $\mathrm{wt}(p)$ of a weighted Motzkin path p is the product of the weights of all steps. Note that the level of an up step or a down step is at least 1 and the level of a horizontal step may be 0.

Let $\mathrm{Mot}_n(a, b)$ denote the set of weighted Motzkin paths of length n such that the weight of an up step of level i is either q^i or -1, the weight of a down step of level i is either abq^{i-1} or -1, and the weight of a horizontal step of level i is either aq^i or bq^i. Then by Viennot's theory [13] we have

$$2^n \mu_n(a, b, 0, 0; q) = \sum_{P \in \mathrm{Mot}_n(a,b)} \mathrm{wt}(P). \tag{4}$$

We define $\mathrm{Mot}^*_n(a, b)$ to be the set of weighted Motzkin paths in $\mathrm{Mot}_n(a, b)$ such that there is no peak of weight 1, that is, an up step of weight -1 immediately followed by a down step of weight -1.

By the same idea that is a variation of Penaud's decomposition as in [8, Proposition 5.1], we have

$$\sum_{P\in \mathrm{Mot}_n(a,b)} \mathrm{wt}(P) = \sum_{k=0}^{n} \left(\binom{n}{\frac{n-k}{2}} - \binom{n}{\frac{n-k}{2}-1} \right) \sum_{P\in \mathrm{Mot}^*_k(a,b)} \mathrm{wt}(P). \tag{5}$$

By (4) and (5), Theorem 3.1 and Corollary 3.2 can be restated as follows.

Theorem 3.3. *We have*

$$\sum_{P\in \mathrm{Mot}^*_k(a,b)} \mathrm{wt}(P) = \sum_{u+v+2t=k} a^u b^v (-1)^t q^{\binom{t+1}{2}} \begin{bmatrix} u+v+t \\ u,v,t \end{bmatrix}_q ,$$

$$\sum_{P\in \mathrm{Mot}^*_k(a,q/a)} \mathrm{wt}(P) = (q/a)^k \sum_{i=0}^{k} a^{2i} q^{i(k-i-1)}.$$

We note that the second identity in Theorem 3.3 is equivalent to a result of Corteel et al. [3, Proposition 5], which is used to prove a formula [3, Theorem 1] for the moments of q-Laguerre polynomials. Their proof of [3, Theorem 1] is combinatorial except for the proof of [3, Proposition 5]. In this section we prove the above theorem combinatorially, thus providing the first combinatorial proof of their result.

In order to give combinatorial proofs of the two formulas in Theorem 3.3 we introduce doubly striped skew shapes. These objects are a generalization of striped skew shapes introduced by D. Kim [11].

A *doubly striped skew shape* of size $m \times n$ is a quadruple (λ, μ, W, B) of partitions $\mu \subset \lambda \subset (n^m)$ and a set W of white stripes and a set B of black stripes with $W \cap B = \emptyset$. Here, a *white stripe* is a diagonal set S of λ/μ such that λ/μ contains neither the cell to the left of the leftmost cells of S nor the cell below the rightmost cell of S, where a diagonal set means a set of cells in row $r+i$ and column $s+i$ for $i = 1, 2, \dots, p$ for some integers r, s, p. Similarly, a *black stripe* is a diagonal set S of λ/μ such that λ/μ contains neither the cell above the leftmost cell of S and the cell to the right of the rightmost cell S. We will call a cell in a white stripe (resp. black stripe) a *white dot* (resp. *black dot*).

Let $\mathrm{DSS}(m, n)$ denote the set of doubly striped skew shapes of size $m \times n$. We define the *weight* of $(\lambda, \mu, W, B) \in \mathrm{DSS}(m, n)$ to be

$$\mathrm{wt}_{a,b}(\lambda, \mu, W, B) = a^m b^n (-1)^{|W|+|B|} q^{|\lambda/\mu| - \|W\| - \|B\|} (q/ab)^{|W|}, \tag{6}$$

where $\|W\|$ and $\|B\|$ are the total numbers of white dots and black dots respectively.

We define a map $\rho : \mathrm{Mot}^*_k(a,b) \to \bigcup_{i=0}^{k} \mathrm{DSS}(i, k-i)$ as follows. Let $P \in \mathrm{Mot}^*_k(a,b)$. We will construct an upper path and a lower path which determine two partitions λ and μ respectively. The upper path and the lower path start at the origin. For each step of P, we add one step to the two lattice paths as follows. If the step is an up step of weight abq^{i-1}, we add a north step of weight a to the upper path and an east step of weight b to the lower path. If the step is an up step of weight -1, we add a north step of weight a to the upper path and an east step of weight b to the lower path, and we make a white stripe between these two steps, see Figure 1. Similarly we add one step to the upper and lower paths for the other types of steps in P as shown in Figure 1. Then we define $\rho(P)$ to be the resulting diagram. See Figure 2 for an example of ρ. It is easy to see that ρ is a weight-preserving bijection, which implies

$$\sum_{P\in \mathrm{Mot}^*_k(a,b)} \mathrm{wt}(P) = \sum_{i=0}^{k} \sum_{S\in \mathrm{DSS}(i,k-i)} \mathrm{wt}_{a,b}(S).$$

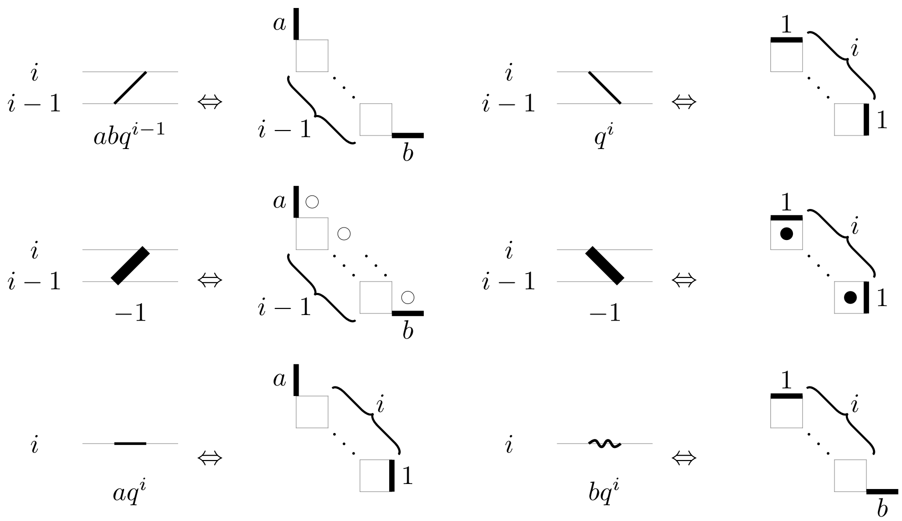

Fig. 1: Converting a weighted Motzkin path to a doubly striped skew shape.

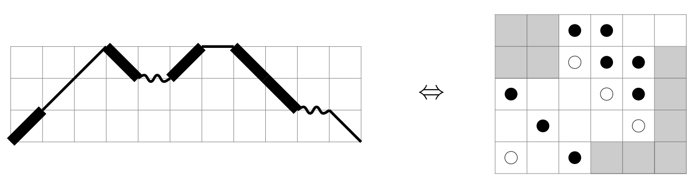

Fig. 2: An example of the bijection ρ. The steps of weight -1 are the thick steps.

Thus Theorem 3.3 is equivalent to the following proposition.

Proposition 3.4. *For $k \geq 0$, we have*

$$\sum_{i=0}^{k} \sum_{S \in \mathrm{DSS}(i,k-i)} \mathrm{wt}_{a,b}(S) = \sum_{u+v+2t=k} a^u b^v (-1)^t q^{\binom{t+1}{2}} \begin{bmatrix} u+v+t \\ u,v,t \end{bmatrix}_q, \tag{7}$$

$$\sum_{i=0}^{k} \sum_{S \in \mathrm{DSS}(i,k-i)} \mathrm{wt}_{a,q/a}(S) = (q/a)^k \sum_{i=0}^{k} a^{2i} q^{i(k-i-1)}. \tag{8}$$

Our proof of Proposition 3.4 is based on D. Kim's sign-reversing involution on striped skew shapes.

4 Matchings

In this section we provide the second combinatorial approach to compute a non-normalized n^{th} moment, using an idea in [7]. The orthogonality relation for Askey-Wilson polynomials is

$$\int_0^\pi P_n(\cos\theta, a, b, c, d; q) P_m(\cos\theta, a, b, c, d; q) w(\cos\theta, a, b, c, d; q) d\theta = 0, \qquad n \neq m.$$

Let

$$I_n := \frac{(q)_\infty}{2\pi} \int_0^\pi (\cos\theta)^n w(\cos\theta, a, b, c, d; q) d\theta.$$

Then $I_n = \mu_n(a, b, c, d; q) I_0$.

The integral I_n, which is a multiple of the n^{th} moment, is the q-exponential generating function for a set of complete matchings. Ismail, Stanton, and Viennot [7] evaluated the Askey-Wilson integral I_0 by interpreting the weight function $w(\cos\theta, a, b, c, d; q)$ as a generating function of four q-Hermite polynomials. By generalizing the method in [7], we obtain the following theorem.

Theorem 4.1. *We have*

$$I_n = \left(\frac{\sqrt{1-q}}{2}\right)^n \sum_{n_1,n_2,n_3,n_4 \geq 0} \frac{\widetilde{a}^{n_1} \widetilde{b}^{n_2} \widetilde{c}^{n_3} \widetilde{d}^{n_4}}{[n_1]_q! [n_2]_q! [n_3]_q! [n_4]_q!} \widetilde{f}_n(n_1, n_2, n_3, n_4; q), \tag{9}$$

where $\widetilde{a} = a/\sqrt{1-q}$, $\widetilde{b} = b/\sqrt{1-q}$, $\widetilde{c} = c/\sqrt{1-q}$, $\widetilde{d} = d/\sqrt{1-q}$, and

$$\widetilde{f}_n(n_1, \dots, n_k; q) = \sum_{\sigma \in \mathcal{CM}(n;n_1,n_2,\dots,n_k)} q^{\mathrm{cr}(\sigma)},$$

where $\mathcal{CM}(n; n_1, n_2, \dots, n_k)$ is the set of complete matchings on $[n] \uplus [n_1] \uplus \cdots \uplus [n_k]$ such that homogeneous edges are contained in $[n]$, see below for the precise definition.

A *matching* is a set partition of $\{1, 2, \dots, n\}$ in which every block has size 1 or 2. A block of size 2 is called an *edge* and a block of size 1 is called a *fixed point*. A *complete matching* is a matching without fixed points. For a matching π we define the *crossing number* $\mathrm{cr}(\pi)$ to be the number of pairs (e_1, e_2) of blocks $e_1, e_2 \in \pi$ such that $e_1 = \{i_1, j_1\}, e_2 = \{i_2, j_2\}$ with $i_1 < i_2 < j_1 < j_2$ or $e_1 = \{i_1, j_1\}, e_2 = \{j_2\}$ with $i_1 < i_2 < j_1$.

For fixed integers $n_0, n_1, n_2, \ldots, n_k$, let $S_i = \{m_{i-1}+1, m_{i-1}+2, \ldots, m_i\}$ for $i = 0, 1, 2, \ldots, k$, where $m_{-1} = 0$ and $m_i = n_0 + n_1 + \cdots + n_i$ for $i = 0, 1, \ldots, k$. We define $\mathcal{CM}(n_0; n_1, n_2, \ldots, n_k)$ to be the set of complete matchings π on $\bigcup_{i=0}^{k} S_i$ such that if an edge of π is contained in S_i, then $i = 0$.

We can use Theorem 4.1 to find new explicit formulas for the moments, and also explain the mixed binomial and q-binomial coefficients. The reason is that the generating function for a crossing number of matchings (not necessarily complete) always has such a formula.

Let $\mathcal{M}(n, m)$ denote the set of matchings on $\{1, 2, \ldots, n\}$ with m fixed points. Note that $\mathcal{M}(n, m) = \emptyset$ unless $n \equiv m \mod 2$. Josuat-Vergès [10, Proposition 5.1] showed the following (see also [2, Proposition 15]): if $n \equiv m \mod 2$, we have

$$\sum_{\pi \in \mathcal{M}(n,m)} q^{\mathrm{cr}(\pi)} = \frac{1}{(1-q)^{(n-m)/2}} \sum_{k \geq 0} \left(\binom{n}{\frac{n-k}{2}} - \binom{n}{\frac{n-k}{2} - 1} \right) (-1)^{(k-m)/2} q^{\binom{(k-m)/2+1}{2}} \begin{bmatrix} \frac{k+m}{2} \\ \frac{k-m}{2} \end{bmatrix}_q . \tag{10}$$

Let

$$P(n, m) = \sum_{\pi \in \mathcal{M}(n,m)} q^{\mathrm{cr}(\pi)}, \qquad \overline{P}(n, m) = (1-q)^{(n-m)/2} P(n, m).$$

An explicit formula for $\widetilde{f}_0(n_1, \ldots, n_k; q)$ in [7, Theorem 3.2] allows one give new formulas for the moments, which are explicit rational functions with the correct singularities.

Theorem 4.2. *We have*

$$2^n \mu_n(a, b, c, d; q) = \sum_{\alpha, \beta, \gamma, \delta \geq 0} a^\alpha b^\beta c^\gamma d^\delta \overline{P}(n, \alpha+\beta+\gamma+\delta) \begin{bmatrix} \alpha+\beta+\gamma+\delta \\ \alpha, \beta, \gamma, \delta \end{bmatrix}_q \frac{(ad)_{\beta+\gamma} (ac)_\beta (bd)_\gamma}{(abcd)_{\beta+\gamma}},$$

$$2^n \mu_n(a, b, c, 0; q) = \sum_{\alpha, \beta, \gamma \geq 0} a^\alpha b^\beta c^\gamma \overline{P}(n, \alpha+\beta+\gamma) \begin{bmatrix} \alpha+\beta+\gamma \\ \alpha, \beta, \gamma \end{bmatrix}_q (ac)_\beta,$$

$$2^n \mu_n(a, b, 0, 0; q) = \sum_{\alpha, \beta \geq 0} a^\alpha b^\beta \overline{P}(n, \alpha+\beta) \begin{bmatrix} \alpha+\beta \\ \alpha \end{bmatrix}_q .$$

Theorem 4.3. *We have*

$$\begin{aligned} 2^n \mu_n(a, b, c, d; q) = \sum_{k=0}^{n} \left(\binom{n}{\frac{n-k}{2}} - \binom{n}{\frac{n-k}{2} - 1} \right) &\sum_{\alpha+\beta+\gamma+\delta+2t=k} a^\alpha b^\beta c^\gamma d^\delta \frac{(ac)_\beta (bd)_\gamma}{(abcd)_{\beta+\gamma}} \\ &\times (-1)^t q^{\binom{t+1}{2}} \begin{bmatrix} \alpha+\beta+\gamma+t \\ \alpha \end{bmatrix}_q \begin{bmatrix} \beta+\gamma+\delta+t \\ \beta, \gamma, \delta+t \end{bmatrix}_q \begin{bmatrix} \delta+\alpha+t \\ \delta \end{bmatrix}_q , \end{aligned}$$

in the second sum, $\alpha, \beta, \gamma, \delta \geq 0$ *and* $-k \leq t \leq k/2$.

In Theorem 4.3, if $c = 0$, then $\gamma = 0$ in the sum. Thus we get Theorem 2.3.

When $ac = bd = q$ in Theorem 4.3, the formula can be simplified and we obtain a positivity theorem of the moments.

Corollary 4.4. *We have*

$$2^n\mu_n(a,b,q/a,q/b;q) = \sum_{k=0}^{n}\left(\binom{n}{\frac{n-k}{2}} - \binom{n}{\frac{n-k}{2}-1}\right)\frac{1}{[k+1]_q}\sum_{\substack{|A|+|B|\le k\\ A+B\equiv k \mod 2}} a^A b^B q^{\frac{k-A-B}{2}},$$

where the second sum is over all integers A and B such that $|A|+|B|\le k$ and $A+B\equiv k \mod 2$. Thus $[n+1]_q!2^n\mu_n(a,b,q/a,q/b;q)$ is a Laurent polynomial in a and b whose coefficients are positive polynomials in q.

Our proof of Corollary 4.4 involves hypergeometric series summations. Since the formula is simple it will be very interesting to find a combinatorial proof of it.

Problem 1. Find a combinatorial proof of Corollary 4.4.

The Laurent polynomiality in Corollary 4.4 seems to be generalized further.

Conjecture 2. $2^n[n+i+j-1]_q!\mu_n(a,b,q^i/a,q^j/b;q)$ is a Laurent polynomial in a,b and polynomial in q with nonnegative coefficients.

5 Staircase tableaux

In this section we review the third combinatorial model, called staircase tableaux, for the moments of Askey-Wilson polynomials. The staircase tableaux were first introduced in [5] and further studied in [4]. Using the staircase tableaux we shall find the coefficient of the first few highest terms in $2^n(abcd;q)_n\mu_n(a,b,c,d;q)$.

A *staircase tableau of size* n is a filling of the Young diagram of the staircase partition $(n,n-1,\dots,1)$ with $\alpha,\beta,\gamma,\delta$ such that every diagonal cell is nonempty, all cells above an α or γ in the same column are empty, and all cells to the left of a β or δ in the same row are empty. Here a *diagonal cell* is a cell in the ith row and $(n+1-i)$th column for some $i\in\{1,2,\dots,n\}$. We denote by $\mathcal{T}(n)$ the set of staircase tableaux of size n.

Each empty cell s of $T\in\mathcal{T}(n)$ is labeled uniquely as follows. Here, for brevity let $\mathrm{right}(s)$ be the first nonempty cell to the right of s in the same row, and $\mathrm{below}(s)$ the first nonempty cell below s in the same column. If $\mathrm{right}(s)$ has a β, then s is labeled with u. If $\mathrm{right}(s)$ has a δ, then s is labeled with q. If $\mathrm{right}(s)$ has an α or γ, and $\mathrm{below}(s)$ has an α or δ, then s is labeled with u. If $\mathrm{right}(s)$ has an α or γ, and $\mathrm{below}(s)$ has an β or γ, then s is labeled with q. See Figure 3 for an example of a staircase tableau and the labeling of its empty cells.

For $T\in\mathcal{T}(n)$, we define $\mathrm{b}(T)$ to be the number of α's and δ's on the diagonal cells, and $A(T)$, $B(T)$, $C(T)$, $D(T)$, $E(T)$ to be the number of α's, β's, γ's, δ's, empty cells labeled with q in T respectively. For example, if T is the staircase tableau in Figure 3, we have $\mathrm{b}(T)=3$, $A(T)=2$, $B(T)=3$, $C(T)=3$, $D(T)=3$, and $E(T)=11$.

Corteel et al. [4] showed that

$$\mu_n(a,b,c,d;q) = \frac{(1-q)^n}{2^n i^n \prod_{j=0}^{n-1}(\alpha\beta-\gamma\delta q^j)} Z_n(-1;\alpha,\beta,\gamma,\delta;q), \tag{11}$$

		β				γ
	γ			α	α	
				δ		
	δ		γ			
		β				
	δ					
β						

u	u	β	q	u	u	γ
q	γ	q	q	α	α	
q	q	q	q	δ		
q	δ	q	γ			
u	u	β				
q	δ					
β						

Fig. 3: A staircase tableau and the labeling of its empty cells.

where $\alpha = \frac{1-q}{(1+ai)(1+ci)}, \beta = \frac{1-q}{(1-bi)(1-di)}, \gamma = \frac{ac(1-q)}{(1+ai)(1+ci)}, \delta = \frac{bd(1-q)}{(1-bi)(1-di)}$, and

$$Z_n(y;\alpha,\beta,\gamma,\delta;q) = \sum_{T\in\mathcal{T}(n)} y^{\mathrm{b}(T)}\alpha^{A(T)}\beta^{B(T)}\gamma^{C(T)}\delta^{D(T)}q^{E(T)}.$$

Since $\alpha\beta - \gamma\delta q^j = \frac{(1-q)^2(1-abcdq^j)}{(1+ai)(1+ci)(1-bi)(1-di)}$, we can rewrite (11) as follows.

Proposition 5.1. *The Askey-Wilson moments satisfy*

$$\begin{aligned} 2^n(abcd;q)_n\mu_n(a,b,c,d;q) = i^{-n} \sum_{T\in\mathcal{T}(n)} (-1)^{\mathrm{b}(T)}(1-q)^{A(T)+B(T)+C(T)+D(T)-n}q^{E(T)} \\ \times (ac)^{C(T)}(bd)^{D(T)}\big((1+ai)(1+ci)\big)^{n-A(T)-C(T)}\big((1-bi)(1-di)\big)^{n-B(T)-D(T)}. \end{aligned}$$

The highest degree term appearing in Proposition 5.1 is $a^nb^nc^nd^nq^{\binom{n}{2}}$. By analyzing staircase tableaux we obtain the coefficients of the first few highest degree terms.

Theorem 5.2. *We have*

$$\begin{aligned} \left[a^nb^nc^nd^nq^{\binom{n}{2}}\right] 2^n(abcd;q)_n\mu_n(a,b,c,d;q) &= \mathrm{Cat}\left(\frac{n}{2}\right), \\ \left[a^{n-1}b^nc^nd^nq^{\binom{n}{2}}\right] 2^n(abcd;q)_n\mu_n(a,b,c,d;q) &= -\,\mathrm{Cat}\left(\frac{n+1}{2}\right), \\ \left[a^{n-1}b^{n-1}c^nd^nq^{\binom{n}{2}}\right] 2^n(abcd;q)_n\mu_n(a,b,c,d;q) &= \mathrm{Cat}\left(\frac{n+2}{2}\right) - \mathrm{Cat}\left(\frac{n}{2}\right), \end{aligned}$$

where $\mathrm{Cat}(n) = \frac{1}{n+1}\binom{2n}{n}$ *if n is a nonnegative integer, and* $\mathrm{Cat}(n) = 0$ *otherwise.*

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