On Orbits of Order Ideals of Minuscule Posets

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Abstract. An action on order ideals of posets considered by Fon-Der-Flaass is analyzed in the case of posets arising from minuscule representations of complex simple Lie algebras. For these minuscule posets, it is shown that the Fon-Der-Flaass action exhibits the cyclic sieving phenomenon, as defined by Reiner, Stanton, and White. A uniform proof is given by investigation of a bijection due to Stembridge between order ideals of minuscule posets and fully commutative Weyl group elements. This bijection is proven to be equivariant with respect to a conjugate of the Fon-Der-Flaass action and an arbitrary Coxeter element.

If P is a minuscule poset, it is shown that the Fon-Der-Flaass action on order ideals of the Cartesian product $P \times [2]$ also exhibits the cyclic sieving phenomenon, only the proof is by appeal to the classification of minuscule posets and is not uniform.

Résumé. Une action sur des idéaux d'ordre d'ensembles partiellement ordonnés, qui ont été considérés par Fon-Der-Flaass, est analysée dans le cas des ensembles ordonnés qui proviennent des représentations minuscules d'algèbres de Lie simples complexes. À propos de ces ensembles ordonnés minuscules, il est démontré que l'action Fon-Der-Flaass offre le phénomène du crible cyclique, tel que défini par Reiner, Stanton et White. Une preuve uniforme est donnée par une étude d'une bijection due à Stembridge entre les idéaux d'ordre d'ensembles ordonnés minuscules et les éléments complètement commutatifs du groupe de Weyl. Il est démontré que cette bijection est équivariante en ce qui concerne un conjugué de l'action Fon-Der-Flaass et un élément de Coxeter arbitraire.

Si P est un ensemble ordonné minuscule, il est démontré que l'action Fon-Der-Flaass sur des idéaux d'ordre du produit cartésien $P \times [2]$ manifeste le phénomène du crible cyclique aussi, mais la preuve de se fait est par appel à la classification des ensembles ordonnés minuscules et n'est pas uniforme.

Keywords: order ideals, antichains, minuscule posets, minuscule representations, fully commutative elements, cyclic sieving phenomenon

1 Introduction

The Fon-Der-Flaass action on order ideals of a poset has been the subject of extensive study since it was introduced in its original form on hypergraphs in Duchet (1974). In this extended abstract of Rush and Shi (2012), we identify a disparate collection of posets characterized by properties from representation theory – the minuscule posets – that exhibits consistent behavior under the Fon-Der-Flaass action. We illustrate the commonality via the cyclic sieving phenomenon of Reiner et al. (2004), which provides a unifying framework for organizing combinatorial data on orbits arising from cyclic actions.

If P is a poset, and $J(P)$ is the set of order ideals of P, partially ordered by inclusion, the Fon-Der-Flaass action Ψ maps an order ideal $I \in J(P)$ to the order ideal $\Psi(I)$ whose maximal elements are the minimal elements of $P \setminus I$. Since Ψ is invertible, it generates a cyclic group $\langle\Psi\rangle$ acting on $J(P)$, but the orbit structure is not immediately apparent.

In Reiner et al. (2004), Reiner, Stanton, and White observed many situations in which the orbit structure of the action of a cyclic group $\langle c\rangle$ on a finite set X may be predicted by a polynomial $X(q) \in \mathbb{Z}[q]$.

Definition. The triple $(X, X(q), \langle c\rangle)$ exhibits the *cyclic sieving phenomenon* if, for any integer d, the number of elements x in X fixed by c^d is obtained by evaluating $X(q)$ at $q = \zeta^d$, where n is the order of c on X and ζ is any primitive n^{th} root of unity.

In the case when $X = J(P)$ and c is the Fon-Der-Flaass action, the natural generating function to consider is the rank-generating function for $J(P)$, which we denote by $J(P; q)$. Here the rank of an order ideal $I \in J(P)$ is given by the cardinality $|I|$ (so that $J(P; q) := \sum_{I \in J(P)} q^{|I|}$).

The minuscule posets are a class of posets arising in the representation theory of Lie algebras that enjoy some astonishing combinatorial properties. We give some background.

Let $\mathfrak{g}$ be a complex simple Lie algebra with Weyl group W and weight lattice Λ. There is a natural partial order on Λ called the *root order* in which one weight μ is considered to be smaller than another weight ω if the difference $\omega - \mu$ may be expressed as a positive linear combination of simple roots. If $\lambda \in \Lambda$ is dominant and the only weights occuring in the irreducible highest weight representation V^λ are the weights in the W-orbit $W\lambda$, then λ is called *minuscule*, and the restriction of the root order to the set of weights $W\lambda$ (which is called the *weight poset*) has two alternate descriptions:

- Let W_J be the maximal parabolic subgroup of W stabilizing λ, and let W^J be the set of minimum-length coset representatives for the parabolic quotient W/W_J. Then there is a natural bijection

$$\begin{array}{rcl} W^J & \longrightarrow & W\lambda \\ w & \longmapsto & w_0 w\lambda \end{array}$$

  (where w_0 denotes the longest element of W), and this map is an isomorphism of posets between the *strong Bruhat order* on W restricted to W^J and the root order on $W\lambda$.

- Let P be the poset of join-irreducible elements of the root order on $W\lambda$. Then P is called the *minuscule poset for* λ, P is ranked, and there is an isomorphism of posets between the weight poset and $J(P)$.

If P is minuscule, Proctor showed (Proctor (1984), Theorem 6) that P enjoys what Stanley calls the *Gaussian* property (cf. Stanley (2012), Exercise 25): There exists a function $f : P \to \mathbb{Z}$ such that, for all positive integers m,

$$J(P \times [m]; q) = \prod_{p \in P} \frac{1 - q^{m+f(p)+1}}{1 - q^{f(p)+1}}.$$

This may be verified case-by-case, but it follows uniformly from the *standard monomial theory* of Lakshmibai, Musili, and Seshadri, as is shown in Proctor (1984). Furthermore, all Gaussian posets are ranked, and if P is Gaussian, we may take f to be the rank function of P.

Thus, for all positive integers m, we are led to consider the triple $(X, X(q), \langle\Psi\rangle)$, where $X = J(P \times [m])$, $X(q) = J(P \times [m]; q)$, and P is any minuscule poset. We are at last ready to state the first two of our main results, answering a question of Reiner.

Theorem 1.1. *Let P be a minuscule poset. If $m = 1$, $(X, X(q), \langle\Psi\rangle)$ exhibits the cyclic sieving phenomenon.*

Theorem 1.2. *Let P be a minuscule poset. If $m = 2$, $(X, X(q), \langle\Psi\rangle)$ exhibits the cyclic sieving phenomenon.*

It turns out that the claim analogous to Theorems 1.1 and 1.2 is false for $m = 3$; computations performed by Kevin Dilks[(i)] reveal that when $m = 3$ and P is the minuscule poset $[3] \times [3]$, the triple $(X, X(q), \langle\Psi\rangle)$ does not exhibit the cyclic sieving phenomenon. However, if P belongs to the third infinite family of minuscule posets (see the classification at the end of the introduction), the same triple exhibits the cyclic sieving phenomenon for all positive integers m. This was proved in Rush and Shi (2011) but is omitted here. The rest of this introduction is devoted to a discussion of Theorems 1.1 and 1.2 and a brief summary of our approach to their proofs.

It should be noted that several special cases of Theorem 1.1 already exist in the literature. When P arises from a Lie algebra with root system of type A, for instance, Theorem 1.1 reduces to a result of Stanley (2009) coupled with Theorem 1.1(b) in Reiner et al. (2004), and it is recorded as Theorem 6.1 in Striker and Williams (2012). The case when the root system is of type B turns out to be handled almost identically, and it is recorded as Corollary 6.3 in Striker and Williams (2012). That being said, our theorem is a generalization of these results, and, in relating Theorem 1.1 to a known cyclic sieving phenomenon for finite Coxeter groups (Theorem 1.6 in Reiner et al. (2004)), we expose the Fon-Der-Flaass action to new algebraic lines of attack.

If P is a finite poset, it is shown in Cameron and Fon-Der-Flaass (1995) that the Fon-Der-Flaass action Ψ may be expressed as a product of the involutive generators $\{t_p\}_{p \in P}$ for a larger group acting on the poset of order ideals $J(P)$. For all $p \in P$ and $I \in J(P)$, $t_p(I)$ is obtained by *toggling I at p*, so that $t_p(I)$ is either the symmetric difference $I \Delta \{p\}$, if this forms an order ideal, or just I, otherwise. In Striker and Williams (2012), Striker and Williams named this group the *toggle group*.

On the other hand, there is a natural labeling of the elements of a minuscule poset P by the Coxeter generators $S = \{s_1, s_2, \ldots, s_n\}$ for the Weyl group W, which is given in Stembridge (1996). In particular, if P is a minuscule poset, there exists a labeling of P such that the linear extensions of the labeled poset (which is called a *minuscule heap*) index the reduced words for the fully commutative element of W representing the topmost coset $w_0 W_J$. This labeling is illustrated in Figure 2 and explained more thoroughly in section 5. It has the following important properties.

First, it realizes the poset isomorphism $J(P) \cong W^J$ explicitly. Given an order ideal $I \in J(P)$ and a linear extension $(x_1, x_2, \ldots, x_t)$ of the partial order restricted to the elements of I, if the corresponding sequence of labels is $(i_1, i_2, \ldots, i_t)$, define $\phi(I)$ to be $s_{i_t} \cdots s_{i_2} s_{i_1}$. Then the map $\phi : J(P) \to W^J$ is an order-preserving bijection.

Second, it indicates a correspondence between Coxeter elements in W and sequences of toggles in $G(P)$: The choice of a linear ordering on the Coxeter generators $S = (s_{i_1}, \ldots, s_{i_n})$ yields a choice of the following.

- An element $t_{(i_1,\ldots,i_n)}$ in the toggle group that executes the following sequence of toggles: first toggle at all elements of P labeled by s_{i_n}, in any order; then toggle at all elements of P labeled by $s_{i_{n-1}}$, in any order;...; then toggle at all elements of P labeled by s_{i_2}, and, finally, toggle at all elements of P labeled by s_{i_1}, and

(i) Computed using code in the computer algebra package `Maple`. The authors also thank Dilks for allowing them the use of his code for subsequent computations.

- A *Coxeter element* $c = s_{i_1} s_{i_2} \cdots s_{i_{n-1}} s_{i_n}$ in the Weyl group, which acts on cosets W/W_J by left translation (i.e., $c(wW_J) = cwW_J$), and thus also acts on W^J.

The theorems that reduce Theorem 1.1 to the cyclic sieving result of Reiner et al. (2004) are as follows.

Theorem 1.3. *For any minuscule poset P and any ordering of $S = (s_{i_1}, \ldots, s_{i_n})$, the actions Ψ and $t_{(i_1,\ldots,i_n)}$ are conjugate in $G(P)$.*

Theorem 1.4. *For any minuscule poset P and any ordering of $S = (s_{i_1}, \ldots, s_{i_n})$, if $\phi : J(P) \to W^J$ is the isomorphism described above, then the following diagram is commutative:*

$$\begin{array}{rcl} J(P) & \xrightarrow{\phi} & W^J \\ t_{(i_1,\ldots,i_n)} \downarrow & & c \downarrow \\ J(P) & \xrightarrow{\phi} & W^J. \end{array}$$

To see that these theorems suffice to demonstrate Theorem 1.1, we quote Theorem 1.6 from Reiner et al. (2004).

Theorem 1.5 (Reiner et al. (2004), Theorem 1.6)**.** *Let W be a finite Coxeter group; let S be the set of Coxeter generators, and let J be a subset of S. Let W^J be the set of minimum-length coset representatives, and let $W^J(q) = \sum_{w \in W^J} q^{l(w)}$, where $l(w)$ denotes the length of w. If $c \in W$ is a regular element in the sense of Springer (1974), then $(W^J, W^J(q), \langle c \rangle)$ exhibits the cyclic sieving phenomenon.*

In Theorem 1.5, if W^J is a distributive lattice, then the length function l also serves as a rank function, so $W^J(q)$ is the rank-generating function. Furthermore, if c is a Coxeter element of W, then $c \in W$ is regular (cf. Springer (1974)).

Overviews of the proofs of Theorem 1.3 and Theorem 1.4 are given in section 6; sections 2, 3, 4, and 5 provide the requisite background. We did not manage to adapt the techniques developed in these sections for the proof of Theorem 1.2, so we appealed to the classification of minuscule posets, from which Theorem 1.2 follows after generating function calculations. Proofs (with details suppressed) may be found in Rush and Shi (2012). Full proofs are available in Rush and Shi (2011).

We close the introduction with a description of the three infinite families and two exceptional cases of minuscule posets and the root systems associated to the Lie algebras from which they arise. The following facts are well-known (cf. for instance, Stembridge (1994)).

- For the root systems of the form A_n, there are n possible minuscule weights, which lead to n associated minuscule posets, namely, all those posets of the form $[j] \times [n+1-j]$ such that $1 \leq j \leq n$. Posets of this form are considered to comprise the *first infinite family*.

- For the root systems of the form B_n, there is 1 possible minuscule weight, which leads to 1 associated minuscule poset, namely, $[n] \times [n]/S_2$. Posets of this form are considered to comprise the *second infinite family*.

- For the root systems of the form C_n, there is 1 possible minuscule weight, which leads to 1 associated minuscule poset, namely, $[2n-1]$. Posets of this form already belong to the first infinite family.

- For the root systems of the form D_n, there are 3 possible minuscule weights, which, because two of the minuscule weights lead to the same minuscule poset, only lead to 2 associated minuscule posets, namely, $[n-1] \times [n-1]/S_2$ and $J^{n-3}([2] \times [2])$. Posets of the latter form are considered to comprise the *third infinite family* (it should be clear that posets of the former form already belong to the second infinite family).
- For the root system E_6, there are 2 possible minuscule weights, which, because both minuscule weights lead to the same minuscule poset, only lead to 1 associated minuscule poset, namely, $J^2([2] \times [3])$. This poset is called the *first exceptional case*.
- For the root system E_7, there is 1 possible minuscule weight, which leads to 1 associated minuscule poset, namely, $J^3([2] \times [3])$. This poset is called the *second exceptional case*.

No other root systems admit minuscule weights.

2 The Fon-Der-Flaass Action

In this section, we introduce and analyze the Fon-Der-Flaass action.

Let $P = (X, <)$ be a partially ordered set, and let $J(P)$ be the set of order ideals of P, partially ordered by inclusion. Following the notation of Cameron and Fon-Der-Flaass (1995), for all order ideals $I \in J(P)$, let

$$Z(I) = \{x \in I : y > x \Longrightarrow y \notin I\},$$

and let

$$U(I) = \{x \notin I : y < x \Longrightarrow y \in I\}.$$

Then the *Fon-Der-Flaass action*, which we denote by Ψ, is defined as follows.

Definition 2.1. For all $I \in J(P)$, $\Psi(I)$ is the unique order ideal satisfying $Z(\Psi(I)) = U(I)$.

Remark 2.2. It is clear from Definition 2.1 that Ψ permutes the order ideals of P.

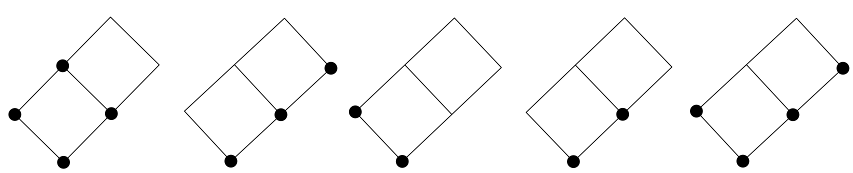

Fig. 1: An orbit of order ideals under the Fon-Der-Flaass action

This definition of the Fon-Der-Flaass action is global. We now give an equivalent definition that decomposes it into a product of local actions, which are more easily understood. Recall from the introduction that for all $p \in P$ and $I \in J(P)$, we let $t_p : J(P) \to J(P)$ be the map defined by $t_p(I) = I \setminus \{p\}$ if $p \in Z(I)$, $t_p(I) = I \cup \{p\}$ if $p \in U(I)$, and $t_p(I) = I$ otherwise. The following theorem is equivalent to Lemma 1 in Cameron and Fon-Der-Flaass (1995).

Theorem 2.3. *Let P be a poset. For all linear extensions $(p_1, p_2, \ldots, p_n)$ of P and order ideals $I \in J(P)$, $\Psi(I) = t_{p_1} t_{p_2} \cdots t_{p_n}(I)$.*

The group $G(P) := \langle t_p \rangle_{p \in P}$ is named the *toggle group* in Striker and Williams (2012). Note that for all x and y, the generators t_x and t_y commute unless x and y share a covering relation.

In the case that the poset P is ranked, it is natural to consider the linear extensions label the elements of P by order of increasing rank. For the purposes of this paper, we shall say that P is ranked if there exists an integer-valued function r on X (called the *rank function*) such that $r(p) = 0$ for all minimal elements $p \in X$ and, for all $x, y \in X$, if x covers y, then $r(x) - r(y) = 1$.

If P is a ranked poset, let the maximum value of r be R. For all $0 \leq i \leq R$, let $P_i = \{p \in P : r(p) = i\}$, and let $t_i = \prod_{p \in P_i} t_p$. We see that t_i is always well-defined because, for all i, t_x and t_y commute for all $x, y \in P_i$. By Theorem 2.3, $\Psi = t_0 t_1 \cdots t_R$. Note that t_i and t_j commute for all $|i - j| > 1$. The following theorem is also a result of Cameron and Fon-Der-Flaass (1995).

Theorem 2.4. *For all permutations σ of $\{0, 1, \ldots, R\}$, $\Psi_\sigma := t_{\sigma(0)} t_{\sigma(1)} \cdots t_{\sigma(R)}$ is conjugate to Ψ in $G(P)$.*

Corollary 2.5. *The action Ψ_σ has the same orbit structure as Ψ for all σ.*

Let $t_{\text{even}} = \prod_{i \text{ even}} t_i$, and let $t_{\text{odd}} = \prod_{i \text{ odd}} t_i$. It should be clear that t_{even} and t_{odd} are well-defined, and it follows from Theorem 2.4 that $t_{\text{even}} t_{\text{odd}}$ is conjugate to Ψ in $G(P)$, as noted in the second paragraph of section 4 in Cameron and Fon-Der-Flaass (1995). This means that the action of toggling at all the elements of odd rank, followed by toggling at all the elements of even rank, is conjugate to the Fon-Der-Flaass action in the toggle group. As we shall see, this holds the key to demonstrating that the induced action of every Coxeter element of W on $J(P)$ under ϕ is conjugate to the Fon-Der-Flaass action as well. Striker and Williams made use of the same argument to obtain the conjugacy of promotion and rowmotion (their name for the Fon-Der-Flaass action) in section 6 of Striker and Williams (2012), so it should be no surprise that our induced actions reduce to promotion in types A and B. In this sense, our proof of Theorem 1.1 may be considered to be a continuation of their work.

3 Minuscule Posets

In this section, we introduce the primary objects of study for this paper – the minuscule posets. We begin with some notation, following Stembridge (1994). Let $\mathfrak{g}$ be a complex simple Lie algebra; let $\mathfrak{h}$ be a Cartan subalgebra; choose a set Φ^+ of positive roots α in $\mathfrak{h}^*$, and let $\Delta = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ be the set of simple roots. Let $(\cdot, \cdot)$ be the inner product on $\mathfrak{h}^*$, and, for each root α, let $\alpha^\vee = 2\alpha/(\alpha, \alpha)$ be the corresponding coroot. Finally, let $\Lambda = \{\lambda \in \mathfrak{h}^* : \alpha \in \Phi \to (\lambda, \alpha^\vee) \in \mathbb{Z}\}$ be the weight lattice.

For all $1 \leq i \leq n$, let s_i be the simple reflection corresponding to the simple root α_i, and let $W = \langle s_i \rangle_{1 \leq i \leq n}$ be the Weyl group of $\mathfrak{g}$. If s is conjugate to a simple reflection s_i in W, we refer to s as an (abstract) reflection.

Let V be a finite-dimensional representation of $\mathfrak{g}$. For each $\lambda \in \Lambda$, let

$$V_\lambda = \{v \in V : h \in \mathfrak{h} \Longrightarrow hv = \lambda(h)v\}$$

be the weight space corresponding to λ, and let Λ_V be the (finite) set of weights λ such that V_λ is nonzero. Recall that there is a standard partial order on Λ called the *root order* defined to be the transitive closure of the relations $\mu < \omega$ for all weights μ and ω such that $\omega - \mu$ is a simple root.

Definition 3.1. The *weight poset* Q_V of the representation V is the restriction of the root order on Λ to Λ_V.

If V is irreducible, Q_V has a unique maximal element, which is called the highest weight of V. This leads to the following definition.

Definition 3.2. Let V be a nontrivial, irreducible, finite-dimensional representation of $\mathfrak{g}$. V is a *minuscule representation* if the action of W on Λ_V is transitive. In this case, the highest weight of V is called the *minuscule weight*.

Theorem 3.3. *If V is minuscule, the weight poset Q_V is a distributive lattice.*

Definition 3.4. If V is minuscule, let P_V be the poset of join-irreducible elements of the weight poset Q_V, so that P_V is the unique poset satisfying $J(P_V) \cong Q_V$. Then P_V is the *minuscule poset* of V, and posets of this form comprise the *minuscule posets*.

Remark 3.5. If V is a minuscule representation and λ is the highest weight of V, we refer to P_V as the *minuscule poset for λ*.

4 Bruhat Posets

In this section, we develop the framework for the proofs of Theorems 1.3 and 1.4. We begin by discussing the Bruhat posets. Then we establish the connection between these objects and the weight posets of minuscule representations.

We continue with the notation of the previous section. Given a Weyl group W, we define a length function l on the elements of W as follows. For all $w \in W$, we let $l(w)$ be the minimum length of a word of the form $s_{i_1}s_{i_2}\dots s_{i_\ell}$ such that $w = s_{i_1}s_{i_2}\dots s_{i_\ell}$ and s_{i_j} is a simple reflection for all $1 \leq j \leq \ell$. This allows us to introduce a well-known partial order on W, known as the (strong) Bruhat order, for which l also serves as a rank function. The Bruhat order is defined to be the transitive closure of the relations $w <_B sw$ for all Weyl group elements w and (abstract) reflections s satisfying $l(w) < l(sw)$.

We are herein concerned not with the Bruhat order on W, but with the restrictions of the Bruhat order to parabolic quotients of W, for these are the orders that give rise to the Bruhat posets.

Definition 4.1. If J is a subset of $\{1, 2, \dots, n\}$, then $W_J := \langle s_i \rangle_{i\in J}$ is the *parabolic subgroup* of W generated by the corresponding simple reflections, and $W^J := W/W_J$ is the *parabolic quotient*.

It is well-known that each coset in W^J has a unique representative of minimum length, so the quotient W^J may be regarded as the subset of W containing only the minimum-length coset representatives. This fact facilitates the definition of an analogous partial order on W^J.

Definition 4.2. The *Bruhat order* $<_B$ on the parabolic quotient W^J is the restriction of the Bruhat order on W to W^J. Posets of the form $(W^J, <_B)$ comprise the *Bruhat posets*.

We may also define the left (weak) Bruhat order on W to be the transitive closure of the relations $w <_L sw$ for all Weyl group elements w and *simple* reflections s satisfying $l(w) < l(sw)$. The analogous partial order on W^J is defined in precisely the same way: $(W^J, <_L)$ is the restriction of $(W, <_L)$ to the minimum-length coset representatives W^J. While the left Bruhat order is not necessary to establish the connection between the minuscule posets and the Bruhat posets, we introduce it here so that our work in this section may be compatible with the theory of fully commutative elements developed in section 5 and exploited in section 6.

We are now ready to state the following theorem, which appears as Proposition 4.1 in Proctor (1984).

Theorem 4.3. *Let V be a minuscule representation with minuscule weight λ, and let $J = \{i : s_i\lambda = \lambda\}$. Then W_J is the stabilizer of λ in the Weyl group W, and the weight poset Q_V is isomorphic to the Bruhat poset $(W^J, <_B)$.*

Definition 4.4. The parabolic quotient W^J is *minuscule* if W_J is the stabilizer of a minuscule weight λ.

The assumption that $\mathfrak{g}$ be simple implies that λ is a fundamental weight. Hence if $\lambda = \omega_j$, then $s_i\lambda = \lambda$ for all $i \neq j$. It follows that if W^J is minuscule, $J = \{1, 2, \ldots, n\} \setminus \{j\}$, so W_J is a maximal parabolic subgroup of W. In general, a minuscule Bruhat poset is obtained precisely when the "missing" element of J is the index of a fundamental weight for which there exists a representation of $\mathfrak{g}$ in which that fundamental weight is minuscule.

We note that Bruhat posets W^J provide a natural setting for identifying instances of the cyclic sieving phenomenon because they come equipped with a group action, namely that of W, and a rank-generating function $W^J(q) := \sum_{w \in W^J} q^{l(w)}$, which is what motivated us to consider them in the first place. We now turn our attention to the labeling of the minuscule poset P_V and the construction of the isomorphism $\phi : J(P_V) \to W^J$, which lie behind the proofs of Theorems 1.3 and 1.4.

5 Fully Commutative Elements

In this section we borrow from Stembridge's theory of fully commutative elements of Weyl groups. In the next section, we shall see how the theory enables us to characterize the relationship between the action of the Weyl group on the elements of these lattices and the action of the toggle group on the order ideals of the corresponding minuscule posets.

Definition 5.1. Let W be a Weyl group, and let $S = \{s_1, s_2, \ldots, s_n\}$ be the set of Coxeter generators. An element $w \in W$ is *fully commutative* if every reduced word for w can be obtained from every other by means of commuting braid relations only (i.e., via relations of the form $s_j s_{j'} = s_{j'} s_j$ for commuting Coxeter generators s_j and $s_{j'}$).

Given a fully commutative element w, we can define a labeled poset P_w that generates all the reduced words of w in the sense that putting labels in the place of poset elements gives a bijection between the linear extensions of P_w and the reduced words of w.

Definition 5.2. Let $s_{i_1} s_{i_2} \cdots s_{i_\ell}$ be a reduced word for w. Let $P_w = (\{1, 2, \ldots, \ell\}, <)$ be a partially ordered set, where the partial order on $\{1, 2, \ldots, \ell\}$ is defined to be the transitive closure of the relations $j > j'$ for all $j < j'$ in integers such that s_{i_j} and $s_{i_{j'}}$ do not commute. Then P_w is the *heap* of w, and, for all $1 \leq j \leq \ell$, s_{i_j} is the *label* of the heap element $j \in P_w$. An example is given in Figure 2.

Let $\mathcal{L}(P_w) := \{\pi : \pi(1) \geq \pi(2) \geq \ldots \geq \pi(\ell)\}$ be the set of *reverse* linear extensions of P_w, and let $\mathcal{L}(P_w, w)$ be the set of labeled reverse linear extensions of P_w, i.e.,

$$\mathcal{L}(P_w, w) := \{s_{i_{\pi(1)}} s_{i_{\pi(2)}} \cdots s_{i_{\pi(\ell)}} : \pi \in \mathcal{L}(P_w)\}.$$

As alluded to above, the set $\mathcal{L}(P_w, w)$ is significant for the following reason.

Proposition 5.3 (Stembridge (1996), Proposition 2.2)**.** *$\mathcal{L}(P_w, w)$ is the set of reduced words for w in W.*

It follows from Proposition 5.3 that, if w is fully commutative, the heaps of the reduced words for w are all equivalent, so we may refer to the heap of w unambiguously. This is also noted in Stembridge (1996). The crucial claim is the next theorem.

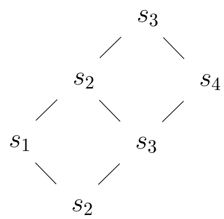

Fig. 2: If W is the Weyl group arising from the root system A_4, then the element $w := s_3s_2s_4s_1s_3s_2$ is fully commutative, and the heap P_w is as displayed above.

Theorem 5.4. *Let $w \in W$ be fully commutative. Then $J(P_w) \cong \{x \in W : x \leq_L w\}$ is an isomorphism of posets.*

We are now ready to define the bijection between $J(P_w)$ and $\{x \in W : x \leq_L w\}$. Given an order ideal $I \in J(P_w)$, let ρ be a linear extension of P_w such that $\rho(j) \in I$ for all $1 \leq j \leq |I|$ and $\rho(j) \notin I$ otherwise.

Theorem 5.5. *The association*

$$I \quad \longmapsto \quad s_{i_{\rho(|I|)}} \cdots s_{i_{\rho(2)}} s_{i_{\rho(1)}}.$$

defines a bijection

$$\phi : J(P_w) \quad \longrightarrow \quad \{x \in W : x \leq_L w\}$$

Remark 5.6. The choice of the symbol ϕ to denote this map is deliberate, for when the heap P_w is minuscule (see Definition 6.1), ϕ is the map described in the introduction.

The following theorem demonstrates the relevance of the theory of fully commutative elements to our main results.

Theorem 5.7 (Stembridge (1996), Theorems 6.1 and 7.1)**.** *If W^J is minuscule, then the following three claims hold:*

(i) If $w \in W^J$, w is fully commutative;

(ii) $(W^J, <_L)$ is a distributive lattice;

(iii) $(W^J, <_B) = (W^J, <_L)$.

6 The Main Results

In this section, we present an overview of our proofs of Theorems 1.3 and 1.4. We start with the following definition and subsequent theorem.

Definition 6.1. If W^J is minuscule, and w_0^J is the longest element of W^J, then the heap $P_{w_0^J}$ is *minuscule*, and heaps of this form comprise the *minuscule heaps*.

Theorem 6.2. *Let V be a minuscule representation of a complex simple Lie algebra $\mathfrak{g}$ with minuscule weight λ and Weyl group W. If $S = \{s_1, s_2, \ldots, s_n\}$ is the set of Coxeter generators and W_J is the maximal parabolic subgroup stabilizing λ, then the following claims hold:*

(i) If w_0^J is the longest element of W^J, then the poset $\{x \in W : x \leq_L w_0^J\}$ and the lattice $(W^J, <_L)$ are identical, and, furthermore, the minuscule heap $P_{w_0^J}$ and the minuscule poset P_V are isomorphic as posets.

(ii) The isomorphism $\phi : J(P_{w_0^J}) \to \{x \in W : x \leq_L w_0^J\} \cong (W^J, <_L) \cong (W^J, <_B)$ defined in Definition 5.5 satisfies the following property: For all $1 \leq k \leq n$, the induced action of the Coxeter generator s_k on $J(P_{w_0^J})$ in the toggle group $G(P_{w_0^J})$ may be expressed in the form $\displaystyle\prod_{\substack{p \in P_{w_0^J} \\ p \text{ is labeled by } s_k}} t_p.$

Example 6.3. In the case when the root system is A_4 and the minuscule weight is ω_2, Figure 3 shows the minuscule heap $P_{s_3s_2s_4s_1s_3s_2}$ (on the left) and the corresponding Bruhat poset $(W^J, <_B)$ (on the right). If I is the order ideal encircled by the solid line, then $\phi(I)$ is the coset representative encircled by the solid line, and $\prod_{p \in P_{s_3s_2s_4s_1s_3s_2} \text{ is labeled by } s_2} t_p(I)$ is the order ideal encircled by the dotted line. Furthermore, $\phi(\prod_{p \in P_{s_3s_2s_4s_1s_3s_2} \text{ is labeled by } s_2} t_p(I)) = s_2\phi(I)$ is the coset representative encircled by the dotted line, thus illustrating the statement *(ii)* in Theorem 6.2.

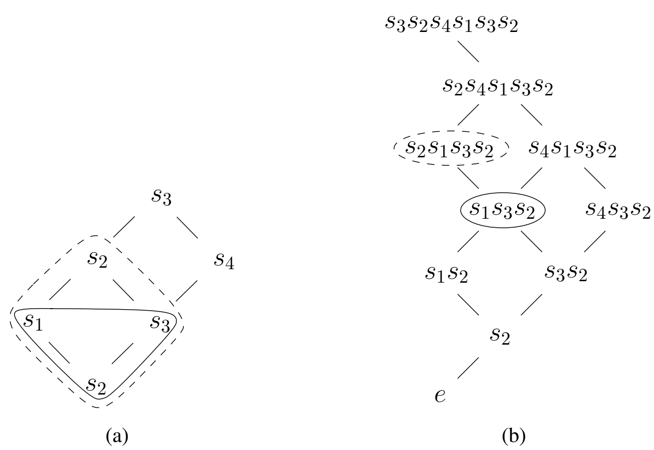

Fig. 3: The map ϕ sends the indicated order ideals to the indicated coset representatives.

We omit the proof of Theorem 6.2, but we will list the three crucial lemmas. For all $1 \leq k \leq n$, let C_k be the set of all heap elements labelled by s_k, and let t'_k be the toggle group element defined by $t'_k = \prod_{p \in C_k} t_p$. From section 5, we know that C_k is totally ordered, and, by definition of $P_{w_0^J}$, no two elements of C_k share a covering relation, so it follows that t'_k is well-defined for all k. The three lemmas are as follows:

Lemma 6.4. *The order ideal $t'_k(I)$ disagrees with I on at most one vertex of $P_{w_0^J}$.*

Lemma 6.5. *There exists an element $p_0 \in P$ such that $p_0 \in Z(I)$ if and only if $s_k w$ is not reduced. In this case, if $s_{i_{l(s_kw)}} \cdots s_{i_2}s_{i_1}$ is a reduced word for s_kw, then $s_k s_{i_{l(s_kw)}} \cdots s_{i_2}s_{i_1}$ is a reduced word for w, and $\phi(I \setminus \{p_0\}) = s_{i_{l(s_kw)}} \cdots s_{i_2}s_{i_1}$.*

Lemma 6.6. *There exists an element $p_0 \in P$ such that $p_0 \in U(I)$ if and only if $s_k w$ is reduced and $s_k w \in W^J$. In this case, if $s_{i_{l(w)}} \cdots s_{i_2} s_{i_1}$ is a reduced word for w, then $s_k s_{i_{l(w)}} \cdots s_{i_2} s_{i_1}$ is a reduced word for $s_k w$, and $\phi(I \cup \{p_0\}) = s_k s_{i_{l(w)}} \cdots s_{i_2} s_{i_1}$.*

Theorem 1.4 is a direct consequence of Theorem 6.2. We proceed to our overview of the proof of Theorem 1.3.

6.1 Proof Overview for Theorem 1.3

Let P_V be a minuscule poset, and again label each element of P_V by the label of the corresponding element of $P_{w_0^J}$. Theorem 6.2 embeds the Weyl group W as a subgroup of the toggle group $G(P_V)$. In light of Theorem 1.4, since the Coxeter elements are known to be pairwise conjugate in W, it suffices to exhibit a particular ordering $S = (s_{i_1}, s_{i_2}, \ldots, s_{i_n})$ such that $t_{(i_1,i_2,\ldots,i_n)} = t'_{i_1} t'_{i_2} \cdots t'_{i_n}$ is conjugate to Ψ in $G(P_V)$. However, in section 2, we saw that $t_{\text{even}} t_{\text{odd}}$ is conjugate to Ψ in $G(P_V)$. It suffices to show that there exists an ordering $(s_{i_1}, s_{i_2}, \ldots, s_{i_n})$ such that the toggle group elements $t'_{i_1} t'_{i_2} \cdots t'_{i_n}$ and $t_{\text{even}} t_{\text{odd}}$ are equal.

The crucial lemmas are the following:

Lemma 6.7 (Stanley (2012), Exercise 25(b))**.** *If P is a minuscule poset, then P is a ranked poset.*

Lemma 6.8 (Björner and Brenti (2005), Chapter 1, Exercise 4)**.** *If W is the Weyl group of a complex simple Lie algebra $\mathfrak{g}$, then the Dynkin diagram of the associated root system is acyclic and therefore bipartite.*

Let r be the rank function for P_V. From these lemmas, we note that if $p, p' \in P_V$ and $r(p) \equiv r(p') \pmod 2$, then t_p and $t_{p'}$ commute in $G(P_V)$. Let S_{odd} be the set of all k such that s_k is a simple reflection and the rank of p is odd for all vertices $p \in P_{w_0^J}$ labelled by s_k. Similarly, let S_{even} be the set of all k' such that $s_{k'}$ is a simple reflection and the rank of p is even for all vertices $p \in P_{w_0^J}$ labelled by $s_{k'}$. It follows that $t_{\text{even}} t_{\text{odd}} = \prod_{k' \in S_{\text{even}}} t'_{k'} \prod_{k \in S_{\text{odd}}} t'_k$.

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References

A. Björner and F. Brenti. *Combinatorics of Coxeter groups*, volume 231 of *Graduate Texts in Mathematics*. Springer, New York, 2005.

P. J. Cameron and D. G. Fon-Der-Flaass. Orbits of antichains revisited. *European J. Combin.*, 16(6): 545–554, 1995.

P. Duchet. Sur les hypergraphes invariantes. *Discrete Math.*, 8:269–280, 1974.

R. A. Proctor. Bruhat lattices, plane partition generating functions, and minuscule representations. *European J. Combin.*, 5(4):331–350, 1984.

V. Reiner, D. Stanton, and D. White. The cyclic sieving phenomenon. *J. Combin. Theory Ser. A*, 108(1): 17–50, 2004.

D. B. Rush and X. Shi. On orbits of order ideals of minuscule posets. 2011. URL `www.math.umn.edu/~reiner/REU/RushShi2011.pdf`.

D. B. Rush and X. Shi. On orbits of order ideals of minuscule posets. *J. Algebraic Combin.,* to appear, 2012. URL `http://www.springerlink.com/content/k2112410603525t5/`.

T. A. Springer. Regular elements of finite reflection groups. *Invent. Math.*, 25:159–198, 1974.

R. P. Stanley. Promotion and evacuation. *Electron. J. Combin.*, 16(2, Special volume in honor of Anders Bjorner):Research Paper 9, 24, 2009.

R. P. Stanley. *Enumerative combinatorics. Volume 1*, volume 49 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, second edition, 2012.

J. R. Stembridge. On minuscule representations, plane partitions and involutions in complex Lie groups. *Duke Math. J.*, 73(2):469–490, 1994.

J. R. Stembridge. On the fully commutative elements of Coxeter groups. *J. Algebraic Combin.*, 5(4): 353–385, 1996.

J. Striker and N. Williams. Promotion and rowmotion. *European J. Combin.*, 33(8):1919–1942, 2012.