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Rainbow supercharacters and a poset analogue to q-binomial coefficients

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Abstract. This paper introduces a variation on the binomial coefficient that depends on a poset and interpolates between q-binomials and 1-binomials: a total order gives the usual q-binomial, and a poset with no relations gives the usual binomial coefficient. These coefficients arise naturally in the study of supercharacters of the finite groups of unipotent upper-triangular matrices, whose representation theory is dictated by the combinatorics of set partitions. In particular, we find a natural set of modules for these groups, whose characters have degrees given by q-binomials, and whose decomposition in terms of supercharacters are given by poset binomial coefficients. This results in a non-trivial family of formulas relating poset binomials to the usual q-binomials.

Résumé. Cet article présente une variation des coefficients binomiaux qui dépend d'un ordre partiel et qui interpole entre les q-binômes et les 1-binômes: un ordre total donne les q-binômes habituelles, et un ordre partiel sans relations donne les coefficients binomiaux habituelles. Ces coefficients apparaissent naturellement dans l'étude des supercaractères de groupes finis de matrices unipotentes, triangulaires supérieures, dont la représentation est dictée par la théorie de la combinatoire des partitions d'ensembles. En particulier, nous trouvons un ensemble naturel de modules pour ces groupes, dont les caractères ont des degrés donnés par les q-binômes, et dont les décompositions en termes de supercaractères sont donnés par les coefficients binômiaux d'ordres partiels. Cela donne une famille non-trivial de formules qui relient les binômes d'ordres partiels aux q-binômes habituelles.

Keywords: q-binomials, posets, supercharacters, set partitions

1 Introduction

The supercharacters of the finite groups of unipotent uppertriangular matrices U_n have seen an increased amount of attention in recent years. While the representation theory of these groups is well-known to be wild, André [2] gave a way to coarsen our notion of an irreducible module in such a way the resulting theory is much more tractable (this notion is generalized to arbitrary finite groups in [5]). More specifically, if $\mathrm{cf}(U_n)$ is the space of class functions of U_n, then André defines a subspace $\mathrm{scf}(U_n)$ which retains many of the nice properties enjoyed by $\mathrm{cf}(U_n)$. Of computational importance is that while the dimension of $\mathrm{cf}(U_n)$ is unknown, the dimension of $\mathrm{scf}(U_n)$ is the nth Bell number.

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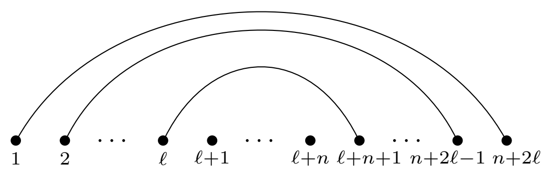
At an American Institute of Mathematics workshop, the participants [1] showed that gluing all the supercharacter theories together

$$\mathrm{scf} = \bigoplus_{n\geq 0} \mathrm{scf}(U_n)$$

gives a Hopf algebra isomorphic to the Hopf algebra of symmetric functions in noncommuting variables. The space scf has two natural bases: one comes from the supercharacters themselves, and certain unions of conjugacy classes (called superclasses) imposed by the supercharacters. The isomorphism in [1] relates the superclass basis to the usual basis of monomial symmetric functions (given, for example, in [6]). While other bases of scf have been studied (such as [3]), the supercharacter basis remains somewhat mysterious in its relation to symmetric functions. One obstruction is that the coproduct – coming from the restriction functor on representations – is not well-understood.

This paper studies the restriction functor on a particular family of supercharacters, which we call "rainbow" supercharacters. They correspond to set partitions of $\{1,2,\ldots,n\}$ of the form $\{\{1,n+2\ell\},\{2,n+2\ell-1\},\ldots,\{\ell,\ell+n+1\},\{\ell+1\},\ldots,\{\ell+n\}\}$ for some $n,\ell\in\mathbb{Z}_{\geq 0}$, which we view as



We are interested in computing the restriction of the corresponding supercharacter to the subgroup

$$U_n \cong U_{[l+1,l+n]} = \left(\begin{array}{c|c|c} \mathrm{Id}_\ell & 0 & 0 \\ \hline 0 & U_n & 0 \\ \hline 0 & 0 & \mathrm{Id}_\ell \end{array}\right) \subseteq U_{2\ell+n}. \tag{1}$$

The restriction problem for these characters is already quite complicated. While there is a recursive algorithm for computing the restriction [7], this algorithm does not give an obvious interpretation for the coefficients.

The main result of this paper observes that these supercharacters in fact restrict to a space

$$\mathbb{Z}_{\geq 0}\text{-span}\{\psi_0^{(n)},\psi_1^{(n)},\ldots,\psi_n^{(n)}\} \subseteq \mathrm{scf}(U_n),$$

spanned by a natural family of characters whose modules are easy to find. Furthermore, they have a combinatorial decomposition in terms of supercharacters with coefficients given by a poset based generalization of q-binomial coefficients. These two facts then give an easy way to compute the coefficients of the restrictions of characters.

The methods from this paper seem to lend themselves to more general statements, and we hope these results will extend to the general restriction problem.

2 Preliminaries

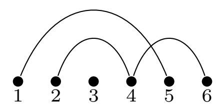
This section reviews our version of set partitions and the particular supercharacter theory that we will use for our main results.

2.1 Set partition combinatorics

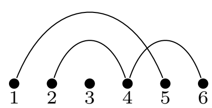
A *set partition* λ *of* $\{1, 2, \ldots, n\}$ is a set of pairs $i \frown j$ with $1 \leq i < j \leq n$ such that if $i \frown j, k \frown l \in \lambda$, then $i = k$ if and only if $j = l$. We typically refer to $i \frown j$ as an *arc* of λ, i as the *left endpoint* of $i \frown j$ and j as the *right endpoint* of $i \frown j$. Let

$$\mathcal{S}_n = \{\text{set partitions of } \{1, 2, \ldots, n\}\}.$$

We typically view these set partitions diagrammatically as a family of arcs on a row of n nodes so that if $i \frown j \in \lambda$, then there is an arc connecting the ith node to the jth node. For example,

$$\lambda = \{1 \frown 5, 2 \frown 4, 4 \frown 6\} \longleftrightarrow$$  .

We can obtain the more traditional version of set partitions as follows. For $\lambda \in \mathcal{S}_n$, define the *blocks* $\mathrm{bl}(\lambda)$ of λ to be the set of equivalence classes on $\{1, 2, \ldots, n\}$ given by the transitive closure of $i \sim j$ if $i \frown j \in \lambda$. For example,

$\mathrm{bl}\Bigg($  $\Bigg) = \{\{1, 5\}, \{2, 4, 6\}, \{3\}\}.$

A *crossing* in a set-partition λ is a pair of arcs $i \frown k, j \frown l \in \lambda$ such that $i < j < k < l$. A *nesting* in a set partition λ is a pair of arcs $i \frown l, j \frown k \in \lambda$ such that $i < j < k < l$. For $\lambda, \mu \in \mathcal{S}_n$, let

$$\begin{aligned} \mathrm{nst}_\mu^\lambda &= \{(i \frown l, j \frown k) \in \lambda \times \mu \mid i < j < k < l\} \\ \mathrm{crs}(\lambda) &= \{(i \frown k, j \frown l) \in \lambda \times \lambda \mid i < j < k < l\}. \end{aligned}$$

Define the set of noncrossing set-partitions to be

$$\mathcal{NCS}_n = \{\lambda \in \mathcal{S}_n \mid \mathrm{crs}(\lambda) = \emptyset\}.$$

Define a function

$$\mathrm{uncr} : \mathcal{S}_n \longrightarrow \mathcal{NCS}_n$$

where $\mathrm{uncr}(\lambda)$ is the unique set partition in $\mathcal{NCS}_n$ that has the same left endpoints and the same right endpoints as λ (in $\mathrm{uncr}(\lambda)$, the first right end-point from the left must be connected to the closest left endpoint, etc.).

Remark 2.1 One can also obtain $\mathrm{uncr}(\lambda)$ from λ by iteratively uncrossing each crossing $\{i \frown k, j \frown l\}$ into a nesting $\{i \frown l, j \frown k\}$. Since this map changes neither the set of left endpoints nor the set of right endpoints, the order in which we "resolve" the crossings does not matter.

2.2 Supercharacters of U_n

A *supercharacter theory* $\mathrm{scf}(G)$ of a finite group G is a subspace of the space of class functions $\mathrm{cf}(G)$ of G such that $\mathrm{scf}(G)$ is a $\mathbb{C}$-subalgebra under the two products

$$\begin{array}{rccc} \circ: & \mathrm{cf}(G)\otimes\mathrm{cf}(G) & \longrightarrow & \mathrm{cf}(G) \\ & \chi\otimes\psi & \mapsto & \begin{array}{rl} \chi\circ\psi: G\to & \mathbb{C} \\ g\mapsto & \displaystyle\frac{1}{|G|}\sum_{h\in G}\chi(h)\overline{\psi(h^{-1}g)} \end{array} \end{array}$$

and

$$\begin{array}{rccc} \odot: & \mathrm{cf}(G)\otimes\mathrm{cf}(G) & \longrightarrow & \mathrm{cf}(G) \\ & \chi\otimes\psi & \mapsto & \begin{array}{rl} \chi\odot\psi: G\to & \mathbb{C} \\ g\mapsto & \chi(g)\psi(g). \end{array} \end{array}$$

Remark 2.2 This definition gives a supercharacter theory as a special kind of S-ring. There is an alternate definition by Diaconis–Isaacs [5] which stresses the partitions of G and $\mathrm{Irr}(G)$ that arise out of the two distinguished bases, described below.

Every such a supercharacter theory has two distinguished bases (up to scalar multiples) given by

$$\begin{aligned} \mathrm{scf}(G) &= \mathbb{C}\text{-span}\{\kappa_A \mid A\in\mathcal{K}\} \\ &= \mathbb{C}\text{-span}\{\sum_{\psi\in B}\psi(1)\psi \mid B\in\mathcal{X}\}, \end{aligned}$$

where $\mathcal{K}$ is a partition of G, $\mathcal{X}$ is a partition of $\mathrm{Irr}(G)$, and for $g\in G$,

$$\kappa_A(g)=\begin{cases} 1 & \text{if } g\in A,\\ 0, & \text{if } g\notin A \end{cases}$$

We typically call the blocks of $\mathcal{K}$ *superclasses*. For each $B\in\mathcal{X}$ we choose constants $c_B\in\mathcal{Q}_{>0}$ such that

$$\chi^B=c_B\sum_{\psi\in B}\psi(1)\psi$$

is a character. The resulting χ^B are called *supercharacters*.

Let U_n be the subgroup of unipotent upper-triangular matrices of the general linear group $\mathrm{GL}_n(\mathbb{F}_q)$ over the finite field $\mathbb{F}_q$ with q elements, and let $T_n\subseteq \mathrm{GL}_n(\mathbb{F}_q)$ be the subgroup of diagonal matrices. While there are many supercharacter theories of U_n, we will focus on the supercharacter theory $\mathrm{scf}(U_n)$ obtained by the following superclasses. Let $u,v\in U_n$ be in the same superclass if there exist $a,b\in U_n$ and $t\in T_n$ such that

$$u=ta(v-\mathrm{Id}_n)bvt^{-1}+\mathrm{Id}_n.$$

Remark 2.3 This supercharacter theory is slightly coarser than the canonical supercharacter theory for algebra groups given by [5], since we let T_n act by outer automorphisms on the usual algebra group supercharacter theory.

The dimension of $\mathrm{scf}(U_n)$ turns out to be Bell number $|\mathcal{S}_n|$. In fact, for every superclass of U_n there exists $\mu \in \mathcal{S}_n$ and a distinguished element u_μ in the superclass such that

$$(u_\mu)_{ij} = \begin{cases} 1, & \text{if } i \frown j \in \mu \text{ or } i = j, \\ 0, & \text{otherwise.} \end{cases}$$

For each $\lambda, \mu \in \mathcal{S}_n$, the supercharacter corresponding to λ is given by

$$\chi^\lambda(\mu) = \chi^\lambda(u_\mu) = \begin{cases} \dfrac{(q-1)^{|\lambda-\mu|} q^{\dim(\lambda)-|\lambda|}(-1)^{|\lambda\cap\mu|}}{q^{\mathrm{nst}^\lambda_\mu}} & \text{if } i \frown l \in \lambda \text{ and } i < j < l \\ & \text{implies } i \frown j, j \frown l \notin \mu, \\ 0 & \text{otherwise,} \end{cases} \tag{2}$$

where

$$\dim(\lambda) = \sum_{i \frown j \in \lambda} j - i.$$

In particular, note that the trivial character $\mathbb{1}$ is the supercharacter associated with the empty partition $\emptyset_n \in \mathcal{S}_n$, and

$$\chi^\lambda(1) = (q-1)^{|\lambda|} q^{\dim(\lambda)-|\lambda|}.$$

Our primary interest is in rainbow characters. That is, for $\ell, n \in \mathbb{Z}_{\geq 0}$, consider

$$\chi^{\{1\frown(2\ell+n), 2\frown(2\ell+n-1), \ldots, \ell\frown\ell+n+1\}},$$

and it follows from the character formula (2), that if U_n is as in (1) and $\mu \in \mathcal{S}_n$, then

$$\begin{aligned} \mathrm{Res}^{U_{2\ell+n}}_{U_n}(\chi^{\{1\frown(2\ell+n), 2\frown(2\ell+n-1), \ldots, \ell\frown(\ell+n+1)\}})(\mu) &= q^{2\binom{\ell}{2}+\ell(n-|\mu|)}(q-1)^\ell \\ &= q^{2\binom{\ell}{2}} \mathrm{Res}^{U_{2\ell+n}}_{U_n}(\underbrace{\chi^{\ell\frown(\ell+n+1)} \odot \cdots \odot \chi^{\ell\frown(\ell+n+1)}}_{\ell \text{ terms}}), \end{aligned}$$

where $\odot$ is the point-wise product on functions. We will therefore focus on the somewhat notationally simpler restriction

$$\mathrm{Res}^{U_{n+2}}_{U_n}(\underbrace{\chi^{1\frown(n+2)} \odot \cdots \odot \chi^{1\frown(n+2)}}_{\ell \text{ terms}}).$$

3 Poset q-Binomials

This section defines the analogue of q-binomials central to this paper. We then restrict our attention to the particular family of posets arising from set partitions that is our focus in the next section.

3.1 *A poset variation on q-binomial coefficients*

Let $\mathcal{P}$ be a poset on n objects. Given a subset $A \subseteq \mathcal{P}$, let

$$\mathrm{wt}(A) = \sum_{a \in A} \mathrm{wt}(a), \qquad \text{where} \qquad \mathrm{wt}(a) = \#\{b \in \mathcal{P} \mid b \succ_{\mathcal{P}} a\}.$$

For $n, k \in \mathbb{Z}_{\geq 0}$, the $\mathcal{P}$-binomial coefficient is

$$\begin{bmatrix} n \\ k \end{bmatrix}_{\mathcal{P}} = \sum_{\substack{S \subseteq \mathcal{P} \\ |S|=k}} q^{\mathrm{wt}(S)}. \tag{3}$$

Note that $\mathrm{wt}(a)$ can also be expressed in terms of the size of the upper ideal containing a.

These coefficients satisfy many recursive relations.

Proposition 3.1 *Let $a \in \mathcal{P}$, $\mathcal{P}' = \mathcal{P} - \{a\}$ and $A = \{a' \mid a' \prec a\}$. Then*

$$\begin{bmatrix} n \\ k \end{bmatrix}_{\mathcal{P}} = \sum_{j=0}^{k} \binom{|A|}{j} q^j \left(q^{\mathrm{wt}(b)} \begin{bmatrix} n-|A|-1 \\ k-j-1 \end{bmatrix}_{\mathcal{P}'} + \begin{bmatrix} n-|A|-1 \\ k-j \end{bmatrix}_{\mathcal{P}'} \right).$$

However, perhaps the most useful one corresponds to removing some minimal element in the poset.

Corollary 3.2 *Let b be a minimal element of a poset $\mathcal{P}$. Let $\mathcal{P}' = \mathcal{P} - \{b\}$. Then*

$$\begin{bmatrix} n \\ k \end{bmatrix}_{\mathcal{P}} = q^{\mathrm{wt}(b)} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_{\mathcal{P}'} + \begin{bmatrix} n-1 \\ k \end{bmatrix}_{\mathcal{P}'}.$$

Examples.

- If $\mathcal{P}$ is the poset with no relations on n elements, then

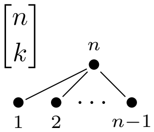
$$\begin{bmatrix} n \\ k \end{bmatrix}_{\mathcal{P}} = \sum_{\substack{S \subseteq \mathcal{P} \\ |S|=k}} q^0 = \binom{n}{k}.$$

- If $\mathcal{P}$ is a total order (say $1 < 2 < \cdots < n$), then

$$\begin{bmatrix} n \\ k \end{bmatrix}_{\mathcal{P}} = \sum_{\substack{S \subseteq \mathcal{P} \\ |S|=k}} q^{\mathrm{wt}(S)} = q^{1+2+\cdots+(k-1)} \sum_{1 \leq a_1 < a_2 < \cdots < a_k \leq n} q^{n-a_1-(k-1)+n-a_2-(k-2)+\cdots+n-a_k-0} = q^{\binom{k}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q.$$

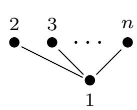
- For the posets with nearly no relations we have

$$\begin{bmatrix} n \\ k \end{bmatrix}_{\mathcal{P}} = \sum_{\substack{S \subseteq \mathcal{P} \\ |S|=k, n \in S}} q^{k-1} + \sum_{\substack{S \subseteq \mathcal{P} \\ |S|=k, n \notin S}} q^k = q^{k-1} \left(\binom{n-1}{k-1} + q \binom{n-1}{k} \right).$$



and

$$\begin{bmatrix} n \\ k \end{bmatrix}_{\mathcal{P}} = \sum_{\substack{S \subseteq \mathcal{P} \\ |S|=k, 1 \in S}} q^{n-1} + \sum_{\substack{S \subseteq \mathcal{P} \\ |S|=k, 1 \notin S}} q^0 = q^{n-1} \binom{n-1}{k-1} + \binom{n-1}{k}.$$



- However, note that in general $\begin{bmatrix} n \\ k \end{bmatrix}_{\mathcal{P}} \neq \begin{bmatrix} n \\ n-k \end{bmatrix}_{\mathcal{P}}$, though the coefficient sequences of the polynomial of one is the reverse coefficient sequence of the other. That is, if

$$b_{\mathcal{P}}(n,k,r,s) = \sum_{\substack{A \sqcup B = \{1,2,\ldots,n\} \\ |A|=k, |B|=n-k}} r^{\mathrm{wt}(A)} s^{\mathrm{wt}(B)},$$

then

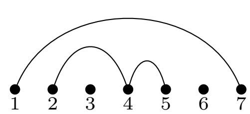
$$\begin{bmatrix} n \\ k \end{bmatrix}_{\mathcal{P}} = b_{\mathcal{P}}(n,k,q,1) \qquad \text{and} \qquad \begin{bmatrix} n \\ n-k \end{bmatrix}_{\mathcal{P}} = b_{\mathcal{P}}(n,k,1,q).$$

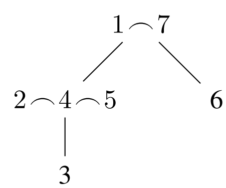
3.1.1 Main example

Let $\lambda \in \mathcal{NCS}_n$. Since there are no crossings in λ, we can define a poset depending on λ based on whether blocks are nested or not. Let $\mathcal{P}(\lambda)$ be the poset on $\mathrm{bl}(\lambda)$ given by $a \prec b$ if either

- $|a| > 1$ and there exist $j, k \in a$ and $i, l \in b$ such that $i < j < k < l$, or
- $a = \{j\}$ and there exist $i, k \in b$ such that $i < j < k$.

For example,

$\lambda =$ 

gives $\mathcal{P}(\lambda) =$ 

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Note that for each $\lambda \in \mathcal{NCS}_n$ the poset $\mathcal{P}(\lambda)$ is a forest where each connected component has a unique maximal element. In fact, all such forests arise in this way (as n and λ vary).

We obtain a function

$$\mathcal{S}_n \xrightarrow{\mathrm{uncr}} \mathcal{NCS}_n \xrightarrow{\mathcal{P}} \left\{ \begin{array}{c} \text{forests where each} \\ \text{connected component has} \\ \text{a unique maximal element} \end{array} \right\}.$$

The following lemma describes a recursion on these binomial coefficients that correspond to small changes in the arcs of the corresponding set partitions (in particular, it allows us to make arcs smaller).

Lemma 3.3 *Fix* $1 \leq i < l-1 < n$. *Let* $\lambda' \in \mathcal{S}_n$ *be such that*

$$\lambda = \lambda' \cup \{i \frown l\} \in \mathcal{S}_n, \quad \mu = \lambda' \cup \{i \frown (l-1)\} \in \mathcal{S}_n, \quad \textit{and} \quad \nu = \lambda' \cup \{i \frown (l-1) \frown l\} \in \mathcal{S}_n.$$

Then for $0 \leq k \leq n - |\lambda|$,

$$q^{\mathrm{nst}_\lambda^\lambda} \begin{bmatrix} n-|\lambda| \\ k \end{bmatrix}_{\mathcal{P}(\mathrm{uncr}(\lambda))} = (q-1) q^{\mathrm{nst}_\nu^\nu} \begin{bmatrix} n-|\nu| \\ k-1 \end{bmatrix}_{\mathcal{P}(\mathrm{uncr}(\nu))} + q^{\mathrm{nst}_\mu^\mu} \begin{bmatrix} n-|\mu| \\ k \end{bmatrix}_{\mathcal{P}(\mathrm{uncr}(\mu))}.$$

3.2 A related module for U_n

Let

$$U_{k\times n} = \bigsqcup_{\substack{A\subseteq\{1,2,\ldots,n\}\\ |A|=k}} \{u_A \mid u \in U_n\},$$

where u_A is the submatrix of u obtained by taking only the rows indexed by A (to obtain a $k\times n$ matrix).

Note that

$$|U_{k\times n}| = \begin{bmatrix} n\\ k\end{bmatrix}_{\mathcal{T}} = q^{\binom{k}{2}}\begin{bmatrix} n\\ k\end{bmatrix}_q,$$

where $\mathcal{T}$ is a total order on n elements. Define

$$V^{k\times n} = \mathbb{C}\text{-span}\{v \mid v \in U_{k\times n}\},$$

which is a right module for U_n under right multiplication.

Remark 3.1 These modules interpolate between the trivial module $V^{0\times n}$ and the regular module $V^{n\times n}$.

Proposition 3.4 *For $\mu \in \mathcal{S}_n$, $0 \leq k \leq n$, and $u \in U_n$ of superclass type μ*

$$\operatorname{tr}(V^{k\times n}, u) = \begin{bmatrix} n-|\mu|\\ k\end{bmatrix}_{\mathcal{T}}.$$

For each $0 \leq k \leq n$, there exists a injection

$$\iota_k : U_{k\times n} \longrightarrow U_n$$

given by $\iota_k(u)$ is the unique matrix obtained by augmenting u with rows consisting of all zeroes except for one 1. For example,

$$\iota_2\begin{pmatrix} 1 & a & b & c & d\\ 0 & 0 & 1 & e & f\end{pmatrix} = \begin{pmatrix} 1 & a & b & c & d\\ 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 1 & e & f\\ 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 1\end{pmatrix}.$$

Let $\theta : \mathfrak{u}_n^* \longrightarrow U_n$ be the bijection given by

$$\theta(\gamma)_{ij} = \begin{cases} \gamma(e_{ij}) & \text{if } 1 \leq i < j < n,\\ 1 & \text{if } i = j,\\ 0 & \text{otherwise,}\end{cases}$$

where e_{ij} is the $n\times n$ matrix with a 1 in the (i,j) entry and zeroes elsewhere. Then the $U_n \times U_n$-orbits in $\mathfrak{u}_n^*$ partition U_n via θ.

Let $\mathfrak{u}_n^*$ be the dual space to the $\mathbb{F}_q$-vector space

$$\mathfrak{u}_n = U_n - \mathrm{Id}_n.$$

The superclasses of U_n are parametrized by $U_n \times U_n$-orbits in $\mathfrak{u}_n$. The supercharacters are parametrized by $(U_n \times U_n) \rtimes T_n$-orbits in $\mathfrak{u}_n^*$, where the action is given by

$$(ta\gamma bt^{-1})(u - \mathrm{Id}_n) = \gamma(a^{-1}t^{-1}(u - \mathrm{Id}_n)tb^{-1}), \qquad \text{for } a, b, u \in U_n, t \in T_n \text{ and } \gamma \in \mathfrak{u}_n^*.$$

For $\lambda \in \mathcal{S}_n$, let $\gamma_\lambda \in \mathfrak{u}_n^*$ be given by

$$\gamma_\lambda(u - \mathrm{Id}_n) = \sum_{i \frown j \in \lambda} u_{ij}.$$

Then each γ_λ is in a different orbit of $\mathfrak{u}_n^*$ (and they therefore are orbit representatives).

For $\lambda \in \mathcal{S}_n$ with $|\lambda| \leq k$, let

$$U_{k\times n}^\lambda = \{u \in U_{k\times n} \mid \theta^{-1}(\iota_k(u)) \in U_n\gamma_\lambda U_n\}.$$

From our partition of U_n given by

$$U_n = \bigsqcup_{\lambda \in \mathcal{S}_n} \mathrm{Id}_n + \theta(U_n\gamma_\lambda U_n),$$

we obtain a partition

$$U_{k\times n} = \bigsqcup_{\substack{\lambda \in \mathcal{S}_n \\ |\lambda| \leq k}} U_{k\times n}^\lambda. \tag{4}$$

The following lemma says how these cardinalities behave as we uncross partitions.

Lemma 3.5 *Let $\{i \frown l, j \frown m\} \in \lambda$ be a crossing with $i < j < l < m$ and $l - j$ maximal. Let $\mu = (\lambda - \{i \frown l, j \frown m\}) \cup \{i \frown m, j \frown l\}$. Then*

$$|U_{k\times n}^\lambda| = \frac{1}{q}|U_{k\times n}^\mu| \qquad \text{and} \qquad q^{\mathrm{nst}_\lambda^\lambda} \begin{bmatrix} n - |\lambda| \\ k - |\lambda| \end{bmatrix}_{\mathcal{P}(\mathrm{uncr}(\lambda))} \chi^\lambda(1) = \frac{1}{q} q^{\mathrm{nst}_\mu^\mu} \begin{bmatrix} n - |\mu| \\ k - |\mu| \end{bmatrix}_{\mathcal{P}(\mathrm{uncr}(\mu))} \chi^\mu(1).$$

We can now give a representation theoretic interpretation for the $\mathcal{P}$-binomials when $\mathcal{P}$ is a forest.

Proposition 3.6 *For $0 \leq k \leq n$ and $\lambda \in \mathcal{S}_n$ with $|\lambda| \leq k$,*

$$|U_{k\times n}^\lambda| = q^{\mathrm{nst}_\lambda^\lambda} \begin{bmatrix} n - |\lambda| \\ k - |\lambda| \end{bmatrix}_{\mathcal{P}(\mathrm{uncr}(\lambda))} \chi^\lambda(1).$$

Observing that the $U_{k\times n}^\lambda$ decompose $U_{k\times n}$, we obtain the following corollary that will be a key step in connecting the poset binomial coefficients to supercharacters.

Corollary 3.7 *For $0 \leq k \leq n$,*

$$\sum_{\lambda \in \mathcal{S}_n} q^{\mathrm{nst}_\lambda^\lambda} \begin{bmatrix} n - |\lambda| \\ k - |\lambda| \end{bmatrix}_{\mathcal{P}(\mathrm{uncr}(\lambda))} \chi^\lambda(1) = \begin{bmatrix} n \\ k \end{bmatrix}_{\mathcal{T}},$$

where $\mathcal{T}$ is a total order on n elements.

4 Decomposition in terms supercharacters of U_n

Corollary 3.7 suggests a possible decomposition for the character of the module $V^{k\times n}$ into supercharacters. Let

$$\psi_k^{(n)} = \sum_{\lambda\in\mathcal{S}_n} q^{\mathrm{nst}_\lambda^\lambda} \begin{bmatrix} n-|\lambda| \\ k-|\lambda| \end{bmatrix}_{\mathcal{P}(\mathrm{uncr}(\lambda))} \chi^\lambda.$$

This section seeks to prove that

$$\psi_k^{(n)}(u) = \mathrm{tr}(V^{k\times n}, u) \qquad \text{for all } u \in U_n.$$

4.1 Main Theorem

The main result is to prove the following simple character formula for the $\psi_k^{(n)}$.

Theorem 4.1 *For $0 \le k \le n$ and $\mu \in \mathcal{S}_n$,*

$$\psi_k^{(n)}(\mu) = \begin{bmatrix} n-|\mu| \\ k \end{bmatrix}_{\mathcal{T}},$$

where $\mathcal{T}$ is a total order on $n-|\mu|$ elements.

Remark 4.1 It would be desirable to have some simple counting argument to prove Theorem 4.1. However, there seems to be some unpredictable cancellation that happens that makes this challenging. For example, for $n=3$ and $k=2$, this theorem implies

$$\binom{3}{2} + \binom{2}{1}(q-1) + \binom{2}{1}(q-1) - (q+1)q + (q-1)^2 = q.$$

We outline the proof by stating three key lemmas, where the proof of the first lemma is the most involved. The basic idea is to show that the $\psi_k^{(n)}(\mu)$ are invariant under adjustments of the arcs in μ. We are then able to reduce the problem to the case when $\mu = \emptyset$, and then use Corollary 3.7.

Let

$$\mathcal{J}_l = \{\lambda \in \mathcal{S}_n \mid l \frown (l+1) \in \lambda\}.$$

Define bijections

$$\varphi_l^R : \mathcal{S}_n - \mathcal{J}_l \longrightarrow \mathcal{S}_n - \mathcal{J}_l \qquad \text{and} \qquad \varphi_l^L : \mathcal{S}_n - \mathcal{J}_l \longrightarrow \mathcal{S}_n - \mathcal{J}_l \tag{5}$$

by letting $\varphi_l^R(\lambda)$ (respectively $\varphi_l^L(\lambda)$) be the set partition obtained from λ by applying the transposition $(l, l+1)$ to all the right (respectively left) endpoints of the arcs in λ. Lemma 4.2 begins by showing that the $\psi_k^{(n)}$ is invariant under small shifts on the arcs.

Lemma 4.2 *Suppose $1 \le l \le n-1$ and $0 \le k \le n$. For $\nu \in \mathcal{S}_n - \mathcal{J}_l$,*

$$\psi_k^{(n)}(\varphi_l^R(\nu)) = \psi_k^{(n)}(\nu) = \psi_k^{(n)}(\varphi_l^L(\nu)).$$

Lemma 4.3 uses Lemma 4.2 to show that $\psi_k^{(n)}$ depends only on the number of arcs associated to a superclass.

Lemma 4.3 *Suppose* $0 \leq k \leq n$ *and* $\mu, \nu \in \mathcal{S}_n$ *with* $|\mu| = |\nu|$. *Then*

$$\psi_k^{(n)}(\mu) = \psi_k^{(n)}(\nu).$$

Lemma 4.4 gives a recurrence on $\psi_k^{(n)}$ in terms of $\psi_k^{(n-1)}$. This allows us to reduce the theorem to evaluating at the identity. Thus, Theorem 4.1 is proved by Corollary 3.7.

Lemma 4.4 *Suppose* $\nu \in \mathcal{S}_n$, $\mu \in \mathcal{S}_{n-1}$ *such that* $|\mu| = |\nu| - 1$. *Then*

$$\psi_k^{(n)}(\nu) = \psi_k^{(n-1)}(\mu).$$

4.2 Some consequences

We first obtain a family of combinatorial identities by applying (2) to Theorem 4.1. For $\lambda \in \mathcal{S}_n$, let

$$\mathrm{cflt}(\lambda) = \{j \frown k \mid 1 \leq j < k < l \text{ with } j \frown l \in \lambda \text{ or } i < j \leq k \leq n \text{ with } i \frown k \in \lambda\}.$$

Note that while the statement of the corollary seems to have negative powers of q, in fact, each summand is a polynomial in q.

Corollary 4.5 *For* $n, k, \in \mathbb{Z}_{\geq 0}$, $\mu \in \mathcal{S}_n$, *and a total order* $\mathcal{T}$ *on* $\{1, 2, \ldots, n - |\mu|\}$,

$$\sum_{\substack{\lambda \in \mathcal{S}_n \\ \mu \cap \mathrm{cflt}(\lambda) = \emptyset}} (-1)^{|\lambda \cap \mu|} \frac{q^{\mathrm{nst}_\lambda^\lambda + \dim(\lambda)}}{q^{\mathrm{nst}_\mu^\lambda + |\lambda|}} (q-1)^{|\lambda - \mu|} \begin{bmatrix} n - |\lambda| \\ k - |\lambda| \end{bmatrix}_{\mathcal{P}(\mathrm{uncr}(\lambda))} = \begin{bmatrix} n - |\mu| \\ k \end{bmatrix}_{\mathcal{T}}.$$

At $q = 1$, this corollary is uninteresting but at $q = 0$ we obtain a sum over compositions of n.

Corollary 4.6 *For* $n, k, \in \mathbb{Z}_{\geq 0}$,

$$\sum_{\lambda \vDash n} (-1)^{n - \ell(\lambda)} \binom{\ell(\lambda)}{\ell(\lambda) - n + k} = \begin{cases} 0 & \text{if } k > 1, \\ 1 & \text{if } k \in \{0, 1\}. \end{cases}$$

However, our main motivation is representation theoretic. By using (2) and a result on q-derivatives of generating functions of q-binomials [4], we obtain the following theorem.

Theorem 4.7

$$\mathrm{Res}^{U_{n+2}}_{U_{[2,n+1]}} (\underbrace{\chi^{1 \frown (n+2)} \odot \cdots \odot \chi^{1 \frown (n+2)}}_{k \text{ terms}}) = (q-1)^k \sum_{j=0}^{k} \left(\prod_{i=0}^{j-1} (q^{k-i} - 1) \right) \psi_j^{(n)},$$

where $U_{[2,n+1]}$ *is as in (1).*

Since we have a decomposition of the $\psi_k^{(n)}$ in terms of supercharacters, we also obtain a formula for the decomposition of the restriction in terms of supercharacters.

Corollary 4.8 *For* $\lambda \in \mathcal{S}_n$,

$$\mathrm{Res}^{U_{n+2}}_{U_{[2,n+1]}} ((\chi^{1 \frown (n+2)})^{\odot k}) = (q-1)^k \sum_{\lambda \in \mathcal{S}_n} \left(\sum_{j=|\lambda|}^{\min(k,n)} q^{\mathrm{nst}_\lambda^\lambda} \begin{bmatrix} n - |\lambda| \\ j - |\lambda| \end{bmatrix}_{\mathcal{P}(\mathrm{uncr}(\lambda))} \prod_{i=0}^{j-1} (q^{k-i} - 1) \right) \chi^\lambda.$$

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