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Spanning forests in regular planar maps

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Abstract. We address the enumeration of p-valent planar maps equipped with a spanning forest, with a weight z per face and a weight u per component of the forest. Equivalently, we count regular maps equipped with a spanning *tree*, with a weight z per face and a weight $\mu := u + 1$ per *internally active* edge, in the sense of Tutte. This enumeration problem corresponds to the limit $q \to 0$ of the q-state Potts model on the (dual) p-angulations.

Our approach is purely combinatorial. The generating function, denoted by $F(z, u)$, is expressed in terms of a pair of series defined by an implicit system involving doubly hypergeometric functions. We derive from this system that $F(z, u)$ is *differentially algebraic*, that is, satisfies a differential equation (in z) with polynomial coefficients in z and u. This has recently been proved for the more general Potts model on 3-valent maps, but via a much more involved and less combinatorial proof.

For $u \geq -1$, we study the singularities of $F(z, u)$ and the corresponding asymptotic behaviour of its n^{th} coefficient. For $u > 0$, we find the standard asymptotic behaviour of planar maps, with a subexponential factor $n^{-5/2}$. At $u = 0$ we witness a phase transition with a factor n^{-3}. When $u \in [-1, 0)$, we obtain an extremely unusual behaviour in $n^{-3}/(\log n)^2$. To our knowledge, this is a new "universality class" of planar maps.

Keywords: Planar maps — Spanning forests — Exact and asymptotic enumeration

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1 Introduction

A *planar map* is a proper embedding of a connected graph in the sphere. The enumeration of planar maps has received a continuous attention since the early 1960s, first in combinatorics with the pioneering work of Tutte, then in theoretical physics, where maps are considered as random surfaces modelling the effect of *quantum gravity*, and more recently in probability theory. General planar maps have been studied, as well as sub-families obtained by imposing constraints of higher connectivity or prescribing the degrees of vertices or faces (*e.g.*, triangulations). Precise definitions are given below.

Several robust enumeration methods have been designed, from Tutte's recursive approach (*e.g.* [25]), which leads to functional equations for generating functions of maps, to the beautiful bijections initiated by Schaeffer [22] and further developed by physicists and combinatorics alike [5, 9], via a powerful approach based on matrix integrals [15]. See for instance [8] for a more complete (though non-exhaustive) bibliography.

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Beyond planar maps, which are now well understood, the attention has also focussed on two more general objects: maps on higher genus surfaces, and maps equipped with an additional structure. The latter question is particularly relevant in physics, where a surface on which nothing happens ("pure gravity") is of little interest. For instance, one has studied maps equipped with a polymer, with an Ising model, with a proper coloring, with a loop model, with a spanning tree, percolation on planar maps... Due to the lack of space, we cannot give the relevant bibliography here.

In particular, several papers have been devoted in the past 20 years to the study of the *Potts model* on families of planar maps [1, 7, 14, 16, 18, 26]. In combinatorial terms, this means counting maps equipped with a colouring in q colours, according to the size (the number of edges) and the number of *monochromatic edges* (edges whose endpoints have the same colour). Up to a change of variables, this also means counting maps weighted by their *Tutte polynomial* (a bivariate combinatorial invariant which has numerous interesting specializations). It has recently been proved that the associated generating function is *differentially algebraic*, that is, satisfies a (non-linear) differential equation (with respect to the size variable) with polynomial coefficients [3, 4, 8]. This holds for general planar maps and for triangulations (or dualy, for cubic maps).

The method that yields these differential equations is extremely involved, and does not shed much light on the structure of q-coloured maps. Moreover, one has not been able, so far, to derive from these equations the asymptotic behaviour of the number of coloured maps, nor the location of phase transitions (see however [7] for recent progress in this direction).

The aim of this paper is to remedy these problems — so far for a one-variable specialization of the Tutte polynomial, obtained by setting to 1 one of its variables, or by taking (in an adequate way) the limit $q \to 0$ in the Potts model. Combinatorially, we are simply counting maps (in this paper, p-valent maps) equipped with a spanning forest. We call them *forested maps* (Figure 1). This problem has already been studied in [12] via a random matrix approach (but with no explicit solution) and, in a special case, in [9], which was in fact the starting point of the present paper. Our enumeration keeps track of the number of faces (variable z) and the number of trees in the forest (minus one; variable u). The case $u = 0$ thus corresponds to maps equipped with a spanning *tree*, solved a long time ago by Mullin [21].

We first obtain in Section 3, in a purely combinatorial manner, a system of functional equations defining the associated generating function $F(z, u)$. We then derive from this system that $F(z, u)$ is differentially algebraic in z, and give explicit differential equations for 3- and 4-valent maps (Section 4). Section 5 is a combinatorial interlude explaining why the series occurring in our system of equations still have non-negative coefficients when $u \in [-1, 0]$. These results are needed in Section 6, which is devoted to asymptotic results: when $u > 0$, forested maps follow the standard asymptotic behaviour of planar maps ($\mu^n n^{-5/2}$) but then a phase transition occurs at $u = 0$, and a very unusual asymptotic behaviour in $\mu^n n^{-3}(\log n)^{-2}$ holds when $u \in [-1, 0)$. To our knowledge, this is the first time a class of planar maps is shown to exhibit this behaviour. This proves in particular that $F(z, u)$ is not *D-finite*, that is, does not

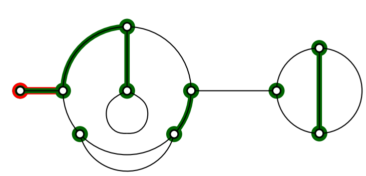

Fig. 1: A (quasi-cubic) forested map with 6 faces and 5 trees.

satisfy any *linear* differential equation in z for $u \in [-1, 0)$ (nor for a generic value of u). This contrasts with the case $u = 0$, for which the generating function of maps equipped with a spanning forest is known to be D-finite.

2 Preliminaries

2.1 Planar maps and trees

A *planar map* is a proper embedding of a connected graph (possibly with loops and multiple edges) in the oriented sphere, considered up to continuous deformation. A *face* is a connected component of the complement of the map. Each edge consists of two *half-edges*, each incident to an endpoint of the edge. A *corner* is an ordered pair (e_1, e_2) of half-edges incident to the same vertex, such that e_2 immediately follows e_1 in counterclockwise order. The *degree* of a vertex is the number of corners incident to it. A vertex of degree p is called *p-valent*. One-valent vertices are also called *leaves*. A map is *p-valent* if all its vertices are p-valent. A *rooted map* is a map with a marked corner (e_1, e_2), indicated by an arrow in our figures. The *root vertex* is the vertex incident to the root. The *root edge* is the edge supporting e_2.

A *(plane) tree* is a planar map with a unique face. A tree is p-valent if all non-leaf vertices have degree p. A *leaf-rooted* (resp. *corner-rooted*) tree is a tree with a marked leaf (resp. corner). A corner-rooted and two leaf-rooted trees appear in Figure 2(b). The number of p-valent leaf-rooted (resp. corner-rooted) trees with k leaves is denoted by t_k (resp. t_k^c). These numbers are well-known [23, Thm. 5.3.10]: they are 0 unless $k = (p-2)\ell + 2$ with $\ell \geq 1$, and in this case,

$$t_k = \frac{((p-1)\ell)!}{\ell!((p-2)\ell+1)!} \qquad \text{and} \qquad t_k^c = p\frac{((p-1)\ell)!}{(\ell-1)!((p-2)\ell+2)!}. \tag{1}$$

These numbers should in principle be denoted $t_{k,p}$ and $t_{k,p}^c$, but we consider p as a fixed integer ($p \geq 3$).

Let M be a rooted planar map with vertex set V. A *spanning forest* of M is a graph $F = (V, S)$ where S is a subset of edges of M forming no cycle. Each tree of F is called a *component*, and the *root component* is the tree containing the root vertex. We say that the pair (M, F) is a *forested map*. Let us denote by $F(z, u)$ the generating function of rooted p-valent forested maps counted by faces (variable z) and non-root components (variable u). For instance, when $p = 3$, the first terms of $F(z, u)$ are

$$F(z,u) = (6 + 4\,u)\,z^3 + \left(140 + 234\,u + 144\,u^2 + 32\,u^3\right) z^4 + \cdots \tag{2}$$

The term $6z^3$ means that there are 6 rooted cubic maps with 3 faces and a distinguished spanning tree.

2.2 Forest counting, the Tutte polynomial and related models

Let $G = (V, E)$ be a graph with vertex set V and edge set E. The *Tutte polynomial* of G is the following polynomial in two indeterminates (see e.g. [6]):

$$\mathrm{T}_G(\mu, \nu) := \sum_{S \subseteq E} (\mu - 1)^{\mathrm{c}(S) - \mathrm{c}(G)} (\nu - 1)^{\mathrm{e}(S) + \mathrm{c}(S) - \mathrm{v}(G)}, \tag{3}$$

where the sum is over all spanning subgraphs of G (equivalently, over all subsets S of edges) and $\mathrm{v}(.)$, $\mathrm{e}(.)$ and $\mathrm{c}(.)$ denote respectively the number of vertices, edges and connected components.

When $\nu = 1$, the only subgraphs that contribute to (3) are the forests. Hence the above defined generating function of forested maps can be written as

$$F(z,u) = \sum_{M\ p-\text{valent}} z^{\mathrm{f}(M)}\, \mathrm{T}_M(u+1,1). \tag{4}$$

Even though this is not clear from (3), the Tutte polynomial $\mathrm{T}_G(\mu,\nu)$ has non-negative coefficients in μ and ν. This was proved combinatorially by Tutte [24], who showed that $\mathrm{T}_G(\mu,\nu)$ counts spanning *trees* of G according to two parameters, called *internal* and *external* activities (see [2] for an alternative description). It follows that $F(z,\mu-1)$ is also the generating function of p-valent planar maps M equipped with a spanning tree T, counted by the face number of M (by z) and the internal activity of T (by μ).

Using duality properties of the Tutte polynomial, and various combinatorial interpretations of $\mathrm{T}_G(1,\nu)$, we can also describe $F(z,u)$ in terms of the dual p-*angulations* equipped: either with a connected (spanning) subgraph; or with a *recurrent configuration of the sandpile model* [13, 20]; or with a q-state Potts model, taken in the limit $q \to 0$. Precise statements and details are given in the complete version of this paper.

2.3 Formal power series

Let $A = A(z) \in \mathbb{K}[[z]]$ be a power series in one variable with coefficients in a field $\mathbb{K}$. We say that A is ***D-finite*** if it satisfies a (non-trivial) linear differential equation with coefficients in $\mathbb{K}[z]$ (the ring of polynomials in z). More generally, it is ***D-algebraic*** if there exist a positive integer k and a non-trivial polynomial $P \in \mathbb{K}[x, x_0, \ldots, x_k]$ such that $P\left(z, A, \frac{\partial A}{\partial z}, \ldots, \frac{\partial^k A}{\partial z^k}\right) = 0$.

A k-variate power series $A = A(z_1, \ldots, z_k)$ with coefficients in $\mathbb{K}$ is ***D-finite*** if its partial derivatives (of all orders) span a finite dimensional vector space over $\mathbb{K}(z_1, \ldots, z_k)$.

3 Generating functions for forested maps

In this section, we give a system of equations that defines the generating function $F(z,u)$ of p-valent forested maps. In fact, it gives an expression of the series $zF'_z(z,u)$ that counts forested maps with a marked face. We also give two simpler systems for two variants of $F(z,u)$, not involving a derivative.

3.1 p-Valent maps

Theorem 3.1 *Fix $p \geq 3$. Let θ, ϕ_1 and ϕ_2 be the following doubly hypergeometric series:*

$$\theta(x,y) := \sum_{i\geq 0}\sum_{j\geq 0} t^{c}_{2i+j} \binom{2i+j}{i,i,j} x^i y^j,$$

$$\phi_1(x,y) := \sum_{i\geq 1}\sum_{j\geq 0} t_{2i+j} \binom{2i+j-1}{i-1,i,j} x^i y^j, \quad \phi_2(x,y) := \sum_{i\geq 0}\sum_{j\geq 0} t_{2i+j+1} \binom{2i+j}{i,i,j} x^i y^j, \tag{5}$$

where t_k and t^c_k are given by (1) *and $\binom{a+b+c}{a,b,c}$ denotes the trinomial coefficient $(a+b+c)!/(a!b!c!)$.*

There exists a unique pair (R,S) of series in z with constant term 0 *and coefficients in $\mathbb{Q}[u]$ that satisfy*

$$R = z + u\,\phi_1(R,S), \tag{6}$$

$$S = u\,\phi_2(R,S). \tag{7}$$

The generating function $F(z,u)$ of p-valent forested maps is characterized by $F(0,u)=0$ and

$$F'_z(z,u)=\theta(R,S). \tag{8}$$

Remarks
1. These equations allow us to compute the first terms of the expansion of $F(z,u)$ in z, for any $p\geq 3$.
2. When p is even, then $t_{2i+1}=0$ for all i and the series S vanishes, which greatly simplifies the system.
3. When $u=0$, an even more drastic simplification occurs: not only $S=0$, but also $R=z$, so that (8) becomes an explicit expression of F', which we can readily integrate:

$$F(z,0)=\sum_{i\geq 0} t^c_{2i}\binom{2i}{i}\frac{z^{i+1}}{i+1}=\sum_{\ell\geq 1}\frac{p((p-1)\ell)!}{(\ell-1)!(1+(p-2)\ell/2)!(2+(p-2)\ell/2)!}z^{2+(p-2)\ell/2}, \tag{9}$$

where we require ℓ to be even if p is odd. This series counts p-valent maps equipped with a spanning tree, and this expression was already proved by Mullin [21, Eq. (5.8)].

In order to prove Theorem 3.1, we first relate $F(z,u)$ to the generating function of planar maps counted by the distribution of their vertex degrees. More precisely, let $M^\diamond \equiv M^\diamond(z,u;g_1,g_2,\ldots;h_1,h_2,\ldots)$ be the generating function of rooted planar maps with a marked face, where u counts non-root vertices, z counts faces, g_k counts non-root vertices of degree k and h_k root vertices (!) of degree k.

Lemma 3.2 *The series $F(z,u)$ is related to $M^\diamond$ through:*

$$zF'_z(z,u)=M^\diamond(z,u;t_1,t_2,\ldots;t^c_1,t^c_2,\ldots). \tag{10}$$

Proof: Start from a p-valent forested map (M,F) and contract each tree of F that is incident to k half-edges (not in F) into a k-valent vertex (Figure 2). This operation can be seen as an extension of Mullin's construction for maps equipped with a spanning tree [21]. It also appears in [12] and in [9, Appendix A], where the authors study 4-valent forested maps such that the root edge is not in the forest.

To recover the forested map (M,F) from the contracted map M', one has to remember, for the root vertex of M', from which corner-rooted tree it came, and for each non-root vertex, from which *leaf-rooted* tree it came (the reason why a leaf-rooted, rather than corner-rooted tree suffices is related to the fact that the rooting of M induces a total order on its half-edges). Each vertex of M' gives a connected component of F. □

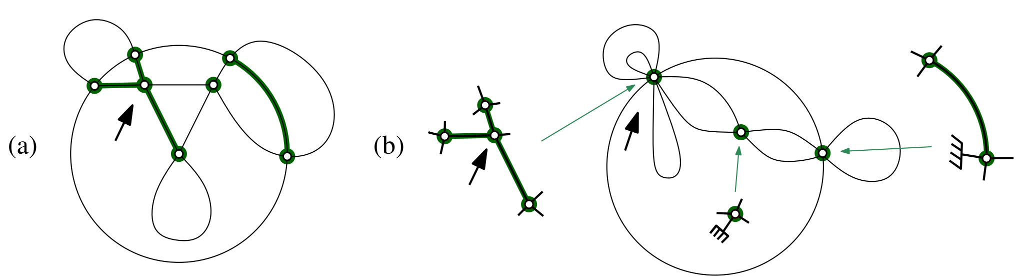

Fig. 2: (a) A 4-valent forested map with 9 faces and 2 non-root components. The arrow indicates the root. (b) The same map, after contraction of the forest, with the collection of rooted trees that stems from (a).

In a recent paper [11, Eq. (2.6)], Bouttier and Guitter have characterized the series $M^\diamond$ by a system of equations, established bijectively. Their system, specialized as in Lemma 3.2, gives Theorem 3.1. $\square$

Remark. In [11, Eq. (1.4)], the authors also give a complicated expression for the generating function $M \equiv M(g_1, g_2, \ldots; h_1, h_2, \ldots)$ that counts rooted planar maps (no marked face) by the distribution of degrees of non-root vertices and the degree of the root vertex. By the above argument, this yields a closed form expression of the series $F(z, u)$ itself. However, we have not been able to use this expression (for instance to construct a differential equation for F) without differentiating it first.

3.2 Two variants

A map is said to be *quasi-p-valent* if all its vertices have degree p, except one vertex that is a leaf. Such maps exist only when p is odd (Figure 1). These maps are naturally rooted at their leaf. Let $G(z, u)$ denote the generating function of quasi-p-valent forested maps counted by faces (z) and non-root components (u).

By relating $G(z, u)$ to the generating function of one-leg maps determined in [10], we obtain:

Proposition 3.3 *The generating function of quasi-p-valent forested maps is*

$$G(z,u) = (1+\bar{u}) \left(zS - u \sum_{i \geq 2} \sum_{j \geq 0} t_{2i+j-1} \binom{2i+j-2}{i-2, i, j} R^i S^j \right),$$

where $\bar{u} = 1/u$, the series R and S are defined by (6–7)*, and the numbers t_k by* (1)*. Also,*

$$G'_z(z,u) = (1+\bar{u})S.$$

We finally consider p-valent forested maps such that the root edge is outside the forest. Let $H(z, u)$ denote the associated generating function. We can relate $H(z, u)$ to the generating function of general planar maps that has been determined in [10]. This yields the following proposition.

Proposition 3.4 *The generating function $H(z, u)$ of p-valent forested maps such that the root edge is outside the forest is*

$$\begin{aligned} H(z,u) &= \bar{u}zR + \bar{u}zS^2 - \bar{u}z^2 \\ &\quad - 2S \sum_{i \geq 2} \sum_{j \geq 0} t_{2i+j-1} \binom{2i+j-2}{i-2, i, j} R^i S^j - \sum_{i \geq 3} \sum_{j \geq 0} t_{2i+j-2} \binom{2i+j-3}{i-3, i, j} R^i S^j \end{aligned}$$

where $\bar{u} = 1/u$, the series R and S are defined by (6–7)*, and the numbers t_k by* (1)*.*

When p is even, then $S = 0$ and the first double sum disappears. In this case, we also have a very simple expression of $H'_z(z, u)$:

$$H'_z(z,u) = 2\bar{u}(R - z).$$

4 Differential equations

The equations established in the previous section imply that series counting regular forested maps are D-algebraic. We prove this and compute explicitly a few differential equations.

Theorem 4.1 *The generating function $F(z,u)$ of p-valent forested maps is D-algebraic (with respect to z). The same holds for the series $G(z,u)$ and $H(z,u)$ of Propositions 3.3 and 3.4.*

Proof: We start from the expression of $F' \equiv F'_z$ given in Theorem 3.1. We first observe that the doubly hypergeometric series θ, ϕ_1, ϕ_2 are D-finite. This follows from the closure properties of D-finite multivariate series [19]. Then, by differentiating (6) and (7) with respect to z, we obtain rational expressions of R' and S' in terms of u and the partial derivatives $\partial\phi_\ell/\partial x$ and $\partial\phi_\ell/\partial y$, evaluated at (R,S) (for $\ell = 1,2$).

Let $\mathbb{K}$ be the field $\mathbb{Q}(u)$. Using (8) and the previous point, it is now easy to prove by induction that for all $k \geq 1$, there exists a rational expression of $F^{(k)}(z,u)$ in terms of

$$\left\{ \frac{\partial^{i+j}\phi_\ell}{\partial x^i \partial y^j}(R,S), \frac{\partial^{i+j}\theta}{\partial x^i \partial y^j}(R,S) \right\}_{i\geq 0, j\geq 0, \ell\in\{1,2\}}$$

with coefficients in $\mathbb{K}$. But since θ, ϕ_1 and ϕ_2 are D-finite, the above set of series spans a vector space of finite dimension, say d, over $\mathbb{Q}(R,S)$. Therefore there exist d elements $\varphi_1,\ldots,\varphi_d$ in this space, and rational functions $A_k \in \mathbb{K}(x,y,x_1,\ldots,x_d)$, such that $F^{(k)}(z,u) = A_k(R,S,\varphi_1,\ldots,\varphi_d)$ for all $k \geq 1$.

Since the transcendence degree of $\mathbb{K}(R,S,\varphi_1,\ldots,\varphi_d)$ over $\mathbb{K}$ is (at most) $d+2$, the $d+3$ series $F', F'', \ldots, F^{(d+3)}$ are algebraically dependant over $\mathbb{K}$. This implies that F' (and thus F) is D-algebraic.

The proof is similar for the series $G(z,u)$ and $H(z,u)$. $\square$

4.1 The 4-valent case

We specialize the above argument to the case $p = 4$. As mentioned below Theorem 3.1, the series S vanishes. The series $F(z,u)$ is characterized by

$$F'_z = \theta(R), \qquad R = z + u\,\phi(R), \tag{11}$$

with

$$\theta(x) = 4\sum_{i\geq 2} \frac{(3(i-1))!}{i!^2(i-2)!}x^i \quad \text{and} \quad \phi(x) = \sum_{i\geq 2} \frac{(3(i-1))!}{i!(i-1)!^2}x^i.$$

The series $\theta(x)$, $\phi(x)$ and their derivatives live in a 3-dimensional vector space over $\mathbb{Q}(x)$ spanned (for instance) by 1, $\theta(x)$ and $\phi(x)$. This follows from the following equations, which are easily checked:

$$x(27x-1)\phi''(x) + 6\phi(x) + 6x = 0, \qquad 3\theta(x) = 2(27x-1)\phi'(x) - 42\phi(x) + 12x.$$

By the argument described above, we can then express F' and all its derivatives as rational functions of u, R, $\phi(R)$ and $\theta(R)$. But since $R = z + u\phi(R)$, this means a rational function of u, z, R and $\theta(R)$. We compute these expressions for F', F'' and F''', eliminate R and $\theta(R)$ from these three equations, and this gives a differential equation of order 2 and degree 7 satisfied by F':

$$\begin{gathered}
9\,F'^2F''^5u^6+36\,F'^2F''^3F'''\,u^5z+144\,F'^2F''^4u^5-12\,(21\,z-1)F'\,F''^5u^5+432\,F'^2F''^2F'''\,u^4z-48\,(24\,z-1)F'\,F''^3F'''\,u^4z\\
+864\,F'^2F''^3u^4-96\,(27\,z-2)F'\,F''^4u^4+4\,(27\,z-1)(15\,z-1)F''^5u^4+1728\,F'^2F''\,F'''\,u^3z-288\,(21\,z-2)F'\,F''^2F'''\,u^3z\\
+10368\,F'\,F'''^2u^2z^3+16\,(27\,z-1)(21\,z-1)F''^3F'''\,u^3z+2304\,F'^2F''^2u^3-288\,(31\,z-4)F'\,F''^3u^3\\
-64\left(6\,uz-162\,z^2+33\,z-1\right)F''^4u^3+2304\,F'^2F'''\,u^2z-2304\,(6\,z-1)F'\,F''\,F'''\,u^2z\\
-192\left(8\,uz-54\,z^2+29\,z-1\right)F''^2F'''\,u^2z-768\,(2\,u+189\,z-7)F'''^2uz^3+2304\,F'^2F''\,u^2-3072\,(3\,z-1)F'\,F''^2u^2\\
-192\left(24\,uz-27\,z^2+55\,z-2\right)F''^3u^2-1536\,(21\,z-2)F'\,F'''\,uz-768\left(12\,uz+81\,z^2+24\,z-1\right)F''\,F'''\,uz+1536\,(9\,z+2)F'\,F''\,u\\
-512\left(39\,uz+81\,z^2+51\,z-2\right)F''^2u+36864\,F'\,z-1024\left(12\,uz-162\,z^2+33\,z-1\right)F'''\,z-1024\,(36\,uz+27\,z-1)F''-24576\,z=0.
\end{gathered}$$

We do not know if F itself satisfies a differential equation of order 2. For the series H of Proposition 3.4 however, a similar approach gives an equation of order 2 and degree 3:

$$3\,(u+1)u^2H'^2H''+12\,u^2zH'H''+6\,(u-8)uH'^2+240\,H \\ +4\,(6\,uz-54\,z+1)H'+4\,(3\,uz^2+30\,uH+27\,z^2-z)H''+24\,z^2=0.$$

4.2 The cubic case

The cubic case is heavier, since we now have to deal with series ϕ_1 and ϕ_2 in two variables. We obtain for $F'(z,u)$ a differential equation of order 2 and degree 17. For the generating function $G(z,u)$ of quasi-cubic forested maps, the degree is only 5:

$$0=(3\,u^4zW'^4-u^3(5\,Wu-uz+z)W'^3+4\,(u+1)(5\,Wu-uz+z)^2)W'' \\ -48\,u^2z(u+1)W'^3+8\,u(u+1)(5\,Wu-uz+z)W'^2+4\,(u-1)(u+1)(5\,Wu-uz+z)W',$$

where $G=(W+z\bar{u})/2$. Introducing the series W is natural in the solution of the Potts model presented in [4], where the above equation was first obtained. It makes the equation more compact.

5 Combinatorics of forested trees

As shown by (4), the series $F(z,u)$ that counts p-valent forested maps has non-negative coefficients in $(1+u)$. We say that it is $(u+1)$-*positive*. More precisely, $F(z,\mu-1)$ counts p-valent maps equipped with a spanning tree weighted by its *internal activity* (by μ). This will lead us to study the asymptotic behaviour of the coefficient of z^n in $F(z,u)$ not only for $u\geq 0$, but for $u\geq -1$. However, our main tool, namely the *singularity analysis* of [17], is much easier when applied to series with non-negative coefficients, and we will need to know that a few other series, related to F, are also $(u+1)$-positive. We prove this thanks to a combinatorial argument that applies to several classes of *forested trees*.

5.1 Positivity in $(1+u)$

Let T be a tree having at least one edge, and $\mathcal{F}$ a set of spanning forests of T. We define a property of $\mathcal{F}$ that guarantees that the generating function $A_{\mathcal{F}}(u)$ that counts forests of $\mathcal{F}$ by the number of components is $(u+1)$-positive (after division by u).

Let $F\in\mathcal{F}$, and let e be an edge of T. By *flipping* e in the forest F, we mean adding e to F if it is not in F, and removing it from F otherwise. This gives a new forest F' on T. We say that e is *flippable* for F if F' still belongs to $\mathcal{F}$. We say that $\mathcal{F}$ is *stable* if for each $F\in\mathcal{F}$ (i) every edge of T not belonging to F is flippable, and (ii) flipping a flippable edge gives a new forest with the same set of flippable edges.

Lemma 5.1 *Assume $\mathcal{F}$ is stable. Then all elements of $\mathcal{F}$ have the same number, say f, of flippable edges, and the generating function of forests of $\mathcal{F}$, counted by components, is $A_{\mathcal{F}}(u)=u(1+u)^f$.*

Proof: The stability of $\mathcal{F}$ implies that the forest $F_{\max}$ consisting of all edges of T belongs to $\mathcal{F}$. Moreover, we can obtain $F_{\max}$ from any forest F of $\mathcal{F}$ by adding iteratively flippable edges. By Condition (ii), this implies that any forest F of $\mathcal{F}$ has the same set of flippable edges as $F_{\max}$. It also means that, to construct a forest F of $\mathcal{F}$, it suffices to choose, for each flippable edge of $F_{\max}$, whether it belongs to F or not. Since $F_{\max}$ has a unique component, and since deleting an edge from a forest increases by 1 the number of components, the expression of $A_{\mathcal{F}}(u)$ follows. □

5.2 Enriched blossoming trees

Define R and S by (6–7), and $\tilde{S}$ by $\tilde{S} = u\,\phi_2(z, \tilde{S})$, where ϕ_2 is given by (5). We now give combinatorial interpretations of these three series in terms of forested trees.

We consider leaf-rooted plane trees, which we draw hanging from their root as in Figure 3. A vertex of degree d is seen as the parent of $d-1$ children. A *subtree* consists of a vertex and all its descendants. A *blossoming tree* is a leaf-rooted plane tree with two kinds of childless vertices: *leaves*, represented by white arrows, and *buds*, represented by black arrows. The edges that carry leaves and buds, as well as the root edge, are considered as half-edges. Each leaf is assigned a *charge* $+1$ while each bud is assigned a charge -1. The *charge* of a subtree is the difference between the number of leaves and buds that it contains.

Definition 5.2 *Let $p \geq 3$. A p-valent blossoming tree equipped with a spanning forest F is an* enriched R- (resp. S-) tree *if (i) its total charge is 1 (resp. 0) and (ii) any subtree rooted at an edge* not in F *has charge 0 or 1. It is an* enriched S̃-tree *if each component of F is incident to as many leaves as buds (in this case it is also an enriched S-tree).*

Proposition 5.3 *The series R, S and $\tilde{S}$ count enriched R-, S- and S̃-trees by the number of leaves (z) and the number of components in the forest (u).*

Proof: The readers who are familiar with the R- and S-trees of [10] will recognize that our enriched R- and S-trees are obtained from them by inflating each vertex of degree k into a (leaf rooted) p-valent tree with k leaves. Thus (6–7) follows from [10] by specializing the indeterminate g_k to t_k.

For the other readers (and for the series $\tilde{S}$), the equations follow from a recursive decomposition of enriched trees. For instance, an enriched R-tree is either reduced to a single leaf, or consists of a root component (say, with k incident edges) in which each non-root incident edge is replaced either by a bud, or an enriched R-tree, or an enriched S-tree. If there are i attached enriched R-trees, we must have $i-1$ buds for the total charge to be 1, and j S-trees with $k-1 = 2i-1+j$. This gives (6). □

Proposition 5.4 *Let T be a p-valent blossoming tree with charge 1 (resp. 0), having at least one edge, and $\mathcal{F}$ the set of spanning forests of T that make it an enriched R- (resp. S-) tree. Then $\mathcal{F}$ is stable, in the sense of Section 5.1. The same holds if T is a p-valent blossoming tree with charge 0, and $\mathcal{F}$ the set of spanning forests of T that make it an enriched S̃-tree.*

Proof: An edge is flippable if and only if the attached subtree has charge 0 or 1. □

By combining this proposition with Lemma 5.1, and Proposition 5.3, we obtain:

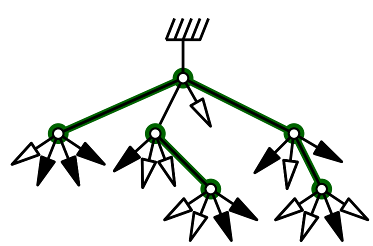

Fig. 3: An enriched 5-valent R-tree. It has 10 leaves (white; charge +1) and 9 buds (black; charge -1).

Corollary 5.5 *The series $\bar{u}(R-z)$, $\bar{u}S$ and $\bar{u}\tilde{S}$ are $(u+1)$-positive.*

6 Asymptotic results

Let $p \geq 3$, and let $F(z,u) = \sum_n f_n(u) z^n$ be the generating function of p-valent forested maps, given by Theorem 3.1. That is, $f_n(u)$ counts p-valent forested maps with n faces by the number of non-root components. As recalled in Section 2.2, $f_n(\mu - 1)$ also counts p-valent maps with n faces equipped with a spanning tree, with a weight μ on each *internally active* edge, or p-angulations equipped with a *recurrent sandpile configuration* weighted (by μ) by its *level* [13, 20]. This explains why we will study the asymptotic behaviour of $f_n(u)$ for any $u \geq -1$.

Here, we first state our results for 4-valent maps and discuss the proof and its difficulties. The fact that p is even simplifies the system of Theorem 3.1 and makes 4-valent maps the most tractable case. We then briefly describe the (analogous) results obtained for cubic maps, with the new difficulties raised by the system of two equations defining the series R and S.

Theorem 6.1 *Let $p = 4$, and take $u \in [-1, +\infty)$. Let $f_n(u)$ be the coefficient of z^n in $F(z,u)$. There exists a positive constant κ_u, depending on u only, such that*

$$f_n(u) \sim \begin{cases} \kappa_u \rho_u^{-n} n^{-3} (\log n)^{-2} & \text{if } u \in [-1, 0), \\ \kappa_u \rho_u^{-n} n^{-3} & \text{if } u = 0, \\ \kappa_u \rho_u^{-n} n^{-5/2} & \text{if } u > 0. \end{cases}$$

Moreover, the radius ρ_u of $F(z,u)$ is an affine function of u when $u \in [-1, 0]$:

$$\rho_u = \frac{1+u}{27} - u\frac{\sqrt{3}}{12\pi}. \qquad (12)$$

For $u > 0$, the subexponential term $n^{-5/2}$ is typical of rooted maps. The behaviour for negative values of u is much more surprising. In fact, we prove that in this case, the singular behaviour of $F'_z(z,u)$ at its unique dominant singularity $\rho \equiv \rho_u$ involves a term $(1 - z/\rho)/\log(1 - z/\rho)$. Since this cannot be the singular behaviour of a D-finite series [17, p. 520 and 582], we have the following corollary.

Corollary 6.2 *For $u \in [-1, 0)$, the generating function $F(z,u)$ of 4-valent forested maps is D-algebraic, but not D-finite. The same holds when u is an indeterminate.*

Note also the simplicity of the radius for $u \leq 0$. In particular, $F(z,-1)$ counts 4-valent maps equipped with a spanning tree having no *internal activity*, and this series has a transcendental radius $\sqrt{3}/(12\pi)$.

Recall that $F(z,0)$ is explicit (see (9)). The estimate of $f_n(0)$ follows from Stirling's formula.

Proof of Theorem 6.1 (sketched): Recall the equations (11) that define the series $F(z,u)$. The key point is to study the singular behaviour of the series R defined implicitly by (11). When $u > 0$, we are in the *smooth implicit function schema* discussed for instance in [17, p. 467]: the series $R(z,u)$ becomes singular when $1 = u\phi'(R)$, and this occurs before R reaches the radius $1/27$ of ϕ and θ. It follows that R, and also F'_z, have a square root singularity at ρ. One then proves that F'_z is analytic in a Δ-domain and concludes, using singularity analysis, that $nf_n(u) \sim \kappa_u \rho_u^{-n} n^{-3/2}$. In brief, the singularities of ϕ and θ are not felt when $u > 0$.

When $u < 0$ however, the conditions of the implicit function theorem do not fail, but R reaches at its radius the singularity of ϕ, namely $1/27$. Since $\phi(1/27) = \sqrt{3}/(12\pi) - 1/27$, Eq. (11) gives the value (12) of ρ_u. The singular behaviour of R now depends on the singular behaviour of ϕ, and is found to be in $(1-z/\rho)/\log(1-z/\rho)$. Similarly, we need to know the singular behaviour of θ to derive the behaviour of F': it is found to behave like R, and we conclude via singularity analysis that $nf_n \sim \kappa_u \rho_u^{-n} n^{-3} (\log n)^{-2}$.

Several ingredients make the case $u < 0$ significantly harder than the case $u > 0$: one of them is that the series R has no longer non-negative coefficients (this is however partially alleviated by Corollary 5.5); it is also harder to prove that R has a unique dominant singularity; finally, we obviously need to know the singular behaviours of ϕ and θ (but these can be found in the literature). $\square$

We have also worked out the asymptotic behaviour of $f_n(u)$ in the cubic case ($p = 3$). This is harder than the 4-valent case, since we now have to deal with a *system* of equations defining R and S (however, $\phi_1(x, y)$ and $\phi_2(x, y)$ can be expressed in terms of hypergeometric functions of $x/(1 - 4y)^2$, which simplifies things a bit). Of course the case $u < 0$ is again harder than the case $u > 0$. Lemma 5.1 and Corollary 5.5 are crucial in the study of this case. The results are the same as in Theorem 6.1, except that the radius for negative u is now a quadratic (rather than affine) function of u, with $\rho_{-1} = \pi^2/384$.

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