FPSAC 2013 Paris, France DMTCS proc. **AS**, 2013, 277–288

The module of affine descent classes of a Weyl group

M. Aguiar[1†] and T. Kyle Petersen[2‡]

[1]*Department of Mathematics, Texas A&M University, College Station, TX, USA*
[2]*Department of Mathematical Sciences, DePaul University, Chicago, IL, USA*

Abstract. The goal of this paper is to introduce an algebraic structure on the space spanned by affine descent classes of a Weyl group, by analogy and in relation to the structure carried by ordinary descent classes. The latter classes span a subalgebra of the group algebra, Solomon's descent algebra. We show that the former span a left module over this algebra. The structure is obtained from geometric considerations involving hyperplane arrangements. We provide a combinatorial model for the case of the symmetric group.

Résumé. Le but de cet article est d'introduire une structure algébrique sur l'espace engendré par les classes de descente affines d'un groupe de Weyl, par rapport à l' structure possédé par les classes de descente finies. Ces dernières engendrent une sous-algèbre de l'algèbre de groupe, l'algèbre de Solomon. Nous montrons que les premières engendrent un module à gauche sur cette algèbre. La structure est obtenue par moyens géométriques impliquant des arrangements d'hyperplans. Un modèle combinatoire est fourni pour le cas du groupe symétrique.

Keywords: Weyl group, Coxeter complex, hyperplane arrangement, Tits product, Solomon's descent algebra, Steinberg torus

1 Introduction

Let W be a finite Coxeter group with simple reflections $S = \{s_1, \dots, s_n\}$ and corresponding simple roots $\Delta = \{\alpha_1, \dots, \alpha_n\}$. For $w \in W$, let $D(w)$ denote the set of *right descents* of w, i.e.,

$$D(w) = \{1 \le i \le n : \ell(w) > \ell(ws_i)\} = \{1 \le i \le n : w\alpha_i < 0\}.$$

For any $J \subseteq [n] := \{1, 2, \dots, n\}$, let

$$x_J := \sum_{D(w) \subseteq J} w$$

denote the sum, in the group ring $\mathbb{Z}W$, of all elements of W whose descent set is contained in J. As J runs over the subsets of $[n]$, the elements x_J span a subring of $\mathbb{Z}W$, denoted $\mathrm{Sol}(W)$, and called *Solomon's descent algebra* (or ring).

†Partially supported by NSF Grant DMS-1001935
‡Partially supported by NSA Grant H98230-10-1-0156

The descent algebra was introduced by Solomon [22] and has been the object of many subsequent works including [2, 3, 4, 5, 7, 9, 12, 13, 15, 17, 20].

The purpose of this paper is to describe a certain left module over Solomon's descent ring. This module is defined in terms of *affine descent sets*, a notion introduced by Cellini [9], and further studied in [10, 11, 18]. This is reviewed next.

We assume that the finite Coxeter group W is *irreducible* and *crystallographic* [14]. In this case, there is a unique *highest* root $\widetilde{\alpha}$ and a corresponding affine Coxeter group $\widetilde{W}$ generated by W and the affine reflection through this highest root (see Section 2.2). Let $\alpha_0 = -\widetilde{\alpha}$. By analogy with ordinary descent sets, the affine descent set, $\widetilde{D}(w)$, of an element $w \in W$, is defined as follows:

$$\widetilde{D}(w) = \{0 \leq i \leq n : w\alpha_i < 0\}.$$

Thus, $D(w) \subseteq \widetilde{D}(w)$, and the only difference occurs when w does not take α_0 to a positive root. Notice that every element has at least one affine descent, and no element can have more than n affine descents.

We emphasize that although the construction of the affine descent module (in Section 3) relies heavily on features of the affine group $\widetilde{W}$, the set $\widetilde{D}(w)$ is defined only for elements w of the finite Coxeter group W, and not for general elements of $\widetilde{W}$.

For any proper nonempty subset J of $\overline{[n]} := \{0, 1, \ldots, n\}$, let

$$\overline{x}_J := \sum_{\widetilde{D}(w) \subseteq J} w.$$

While the elements $\overline{x}_J$ do not span a subring of $\mathbb{Z}W$, we show that they span a left module over $\mathrm{Sol}(W)$. We remark that Cellini showed that the elements $\sum_{|J|=k} \overline{x}_J$, as k runs from 1 to n, do span a subring (in fact, a commutative nonunital subring) of $\mathbb{Z}W$.

We follow the geometric approach of Tits (in his appendix to Solomon's paper [22]), as developed by Bidigare [6] and Brown [8, Section 4.8]. These works relate the algebraic structure of $\mathrm{Sol}(W)$ to the geometric structure of the Coxeter complex. Specifically, the elements x_J correspond to W-orbits of faces in the Coxeter complex. Work of Dilks, Petersen, and Stembridge [10] shows that the $\overline{x}_J$ correspond to W-orbits in the *Steinberg torus*, an object obtained by taking the quotient of the affine Coxeter complex by the co-root lattice.

Here we show that the faces of the Coxeter complex act on the faces of the affine Coxeter complex, and that this action passes through the quotient to an action on the Steinberg torus.

The action on affine faces admits a simple geometric description. An affine hyperplane arrangement splits the ambient space into a set of faces. The hyperplane *at infinity* is similarly decomposed into a set of faces. The latter set is a monoid under the Tits product and the former a right module over it. In the case of the affine arrangement of W, the faces at infinity constitute the Coxeter complex, affine faces are acted upon by co-root translations, and the quotient by this action is the set the faces of the Steinberg torus. It follows that the set of faces of the Steinberg torus is a right module over the Coxeter complex. The structure is equivariant with respect to the Weyl group, and we may consider the induced structure on orbits. This results in the left module structure of affine descent classes over Solomon's descent ring.

Section 2 describes these geometric aspects, providing background on both finite and affine Coxeter complexes and how they can both be viewed inside the closure of the *Tits cone*. We discuss the Steinberg torus as well, and give combinatorial models for faces of all these complexes in Type A_{n-1}. (For the affine Coxeter complex and the Steinberg torus, this model appears to be new.) Section 3 relates the geometric actions to the module structures.

We thank the referees for pointing out the work of Moszkowski concerning modules over Solomon's descent algebra [16]. We plan to explore possible connections to this work and also to that of Saliola [19, 21] in the future. This extended abstract is a condensed version of a longer article; many details and most of the proofs have been omitted.

2 Products of faces in the Tits cone

Let W be a finite Coxeter group with a crystallographic root system Φ embedded in a real Euclidean space V with inner product $\langle \cdot\,, \cdot\rangle$. (So W is a Weyl group.) For any root $\beta \in \Phi$, let $H_\beta := \{\lambda \in V : \langle \lambda, \beta\rangle = 0\}$ be the hyperplane orthogonal to β and let s_β denote the orthogonal reflection through H_β. If we fix a set of simple roots $\Delta = \{\alpha_1, \ldots, \alpha_n\} \subset \Phi$, then $S = \{s_1, \ldots, s_n\}$ denotes the corresponding set of simple reflections.

Having fixed a choice of simple roots, every root β either belongs to the nonnegative span of the simple roots and is designated *positive*, or else belongs to the nonpositive span of the simple roots and is designated *negative*. We write $\beta > 0$ or $\beta < 0$ accordingly. Let $\Pi = \{\beta \in \Phi : \beta > 0\}$ denote the set of positive roots.

2.1 The finite Coxeter complex

The set of hyperplanes $\mathcal{H}(\Phi) := \{H_\beta : \beta \in \Pi\}$ is the *Coxeter arrangement* associated to Φ (we take $\beta > 0$ for convenience since $H_\beta = H_{-\beta}$). A *face* F of the arrangement is any subset of V obtained by choosing, for each $\beta \in \Pi$, either the hyperplane H_β or one of the open half-spaces it bounds, and intersecting all these sets. A face F is determined by its *sign vector*:

$$\sigma(F) = (\sigma_\beta(F))_{\beta\in\Pi},$$

where if λ is any point in F, $\sigma_\beta(F) = +, -$, or 0, according to whether $\langle \lambda, \beta\rangle$ is positive, negative, or zero. Let Σ be the set of faces of $\mathcal{H}(\Phi)$.

We partially order Σ by inclusion of the face closures, i.e., $F \le G \Leftrightarrow \overline{F} \subseteq \overline{G}$. This partial order gives Σ a structure isomorphic to the *Coxeter complex* of W, defined abstractly as the set of cosets of parabolic subgroups of W, ordered by reverse inclusion.

There is a monoid structure on the faces of the Coxeter arrangement, given geometrically as follows. For two faces F and G in Σ, their product FG is the first face of Σ entered upon traveling a small positive distance on a straight line from a point of F to a point in G. See Figure 1. This product is associative and admits the following characterization in terms of sign vectors [1, Proposition 2.82]:

$$\sigma_\beta(FG) = \begin{cases} \sigma_\beta(F) & \text{if } \sigma_\beta(F) \neq 0, \\ \sigma_\beta(G) & \text{if } \sigma_\beta(F) = 0. \end{cases} \tag{1}$$

An alternative characterization of the faces of the Coxeter complex is given in terms of the action of the group W on Σ. The choice of simple roots Δ is equivalent to designating a dominant chamber, namely:

$$C_\emptyset := \{\lambda \in V : \langle \lambda, \alpha\rangle > 0 \text{ for all } \alpha \in \Delta\}.$$

This is the unique face with sign vector $(+, +, \ldots)$. The closure of the dominant chamber is a fundamental domain for the action of W on V, and thus every face has the form wC_J, where $w \in W$, $J \subseteq [n]$, and

$$C_J := \{\lambda \in V : \langle \lambda, \alpha_j\rangle = 0 \text{ for } j \in J, \langle \lambda, \alpha_j\rangle > 0 \text{ for } j \in [n] - J\}.$$

The set J is uniquely determined by the face, but in general the element w is not.

The rays (1-dimensional faces) have the form wC_J where $J = [n] - \{j\}$ for some j. If we assign color j to all such rays, we obtain a *balanced* coloring of Σ; i.e., every maximal face (chamber) has exactly one vertex (extreme ray) of each color. We see that in general a face wC_J has color set $J^c = [n] - J$.

For a positive root β and $\lambda \in C_J$, we have $\langle w\lambda, \beta\rangle = \langle \lambda, w^{-1}\beta\rangle$, so we can characterize sign vectors as follows:

$$\sigma_\beta(wC_J) = \begin{cases} 0 & \text{if } w^{-1}\beta \in \mathrm{Span}\{\alpha_j : j \in J\} \\ + & \text{if } w^{-1}\beta \in \Pi - \mathrm{Span}\{\alpha_j : j \in J\} \\ - & \text{if } -w^{-1}\beta \in \Pi - \mathrm{Span}\{\alpha_j : j \in J\}. \end{cases}$$

In particular, notice that if $w\alpha_i = -\beta < 0$, i.e., if i is a descent of w, then $w^{-1}\beta = -\alpha_i$ and we have $\langle \lambda, -\alpha_i\rangle \leq 0$. Thus the descents of w are encoded among the zeroes and minus signs of $\sigma(wC_J)$. Conversely, if $\sigma_{w\alpha_i}(wC_J) = +$, then i *cannot* be a descent of w.

As seen from (1), the product of a face F with a chamber C always results in another chamber (none of the entries are zero), the *Tits projection* of F onto C.

In particular, we can characterize projections onto the fundamental chamber as follows. Since the fundamental chamber has sign vector $\sigma(C_\emptyset) = (+, +, \ldots)$, Equation (1) gives:

$$\sigma_\beta(FC_\emptyset) = \begin{cases} \sigma_\beta(F) & \text{if } \sigma_\beta(F) \neq 0, \\ + & \text{if } \sigma_\beta(F) = 0. \end{cases}$$

Let w_F denote the unique element of W such that $w_F C_\emptyset = FC_\emptyset$. Since $+$ signs cannot correspond to descents, we have the following.

Proposition 2.1 *For any face F of Σ, $D(w_F) \subseteq \mathrm{col}(F)$. Moreover, for any $w \in W$ and any J with $D(w) \subseteq J \subseteq [n]$, there is a J-colored face F such that $w = w_F$.*

There is a well-known combinatorial model for the Coxeter complex of Type A_{n-1}; see Figure 1. The faces are encoded with *set compositions* of $[n]$, i.e., set partitions with a linear order on the set of blocks. The partial order on faces is given by refinement. The product of two faces is given by refining the first set composition according to the second. For example, $3567|4|12 \cdot 26|35|17|4 = 6|35|7|4|2|1$, where the blocks are separated by bars and the set of blocks is ordered from left to right.

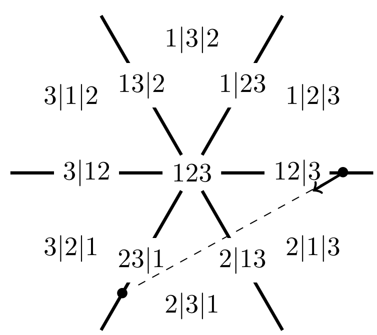

Fig. 1: The product of faces in the Coxeter arrangement of type A_2: $12|3 \cdot 23|1 = 2|1|3$.

The color of a face corresponds to the positions of the vertical bars, and the element w_F is the permutation obtained by writing the elements of the blocks in increasing order and removing the bars. For example, if $F = 3|4|156|2$, $\mathrm{col}(F) = \{1, 2, 5\}$, and $w_F = 341562$. Since the elements of the blocks are written in increasing order, descents can only occur between blocks, i.e., in the locations of the bars.

2.2 The affine Coxeter complex

The affine Weyl group $\widetilde{W}$ is generated by reflections $s_{\beta,k}$ through the affine hyperplanes

$$H_{\beta,k} := \{\lambda \in V : \langle\lambda, \beta\rangle = k\} \qquad (\beta \in \Phi,\ k \in \mathbb{Z}).$$

Alternatively, one may construct $\widetilde{W}$ as the semidirect product $W \ltimes \mathbb{Z}\Phi^\vee$, where $\mathbb{Z}\Phi^\vee$ denotes the lattice generated by all co-roots $\beta^\vee = 2\beta/\langle\beta, \beta\rangle$ ($\beta \in \Phi$). The action of $\widetilde{W}$ on V extends the action of W by linear reflections and the action of $\mathbb{Z}\Phi^\vee$ by translations.

Suppose from now on that Φ is irreducible. Then it has a unique highest root $\widetilde{\alpha}$, and it is well-known that $\widetilde{W}$ is generated by $\widetilde{S} := S \cup \{s_{\widetilde{\alpha},1}\}$ and that $(\widetilde{W}, \widetilde{S})$ is an irreducible Coxeter system.

The *affine Coxeter arrangement* is

$$\widetilde{\mathcal{H}}(\Phi) := \{H_{\beta,k} : \beta \in \Pi, k \in \mathbb{Z}\}.$$

The set of faces of $\widetilde{\mathcal{H}}(\Phi)$ is isomorphic to the affine Coxeter complex of $\widetilde{W}$. We denote it by $\widetilde{\Sigma}$.

A face can again be encoded by a sign vector that records whether the face is "above", "below", or "on" a particular hyperplane. (That the sign vector is now infinite is not a problem. See [1, Section 2.7].) We have $\sigma(F)$, for F a nonempty face in $\widetilde{\Sigma}$, given by

$$\sigma(F) = (\sigma_{\beta,k}(F))_{\beta\in\Pi,k\in\mathbb{Z}}, \tag{2}$$

where $\sigma_{\beta,k}(F)$ is $+, -$, or 0, according to whether $\langle\lambda, \beta\rangle - k$ is positive, negative, or zero. Notice, however, that for a given positive root β, we have a unique j such that

$$(\ldots, \sigma_{\beta,-1}(F), \sigma_{\beta,0}(F), \sigma_{\beta,1}(F), \ldots) = (\ldots, +, +, \sigma_{\beta,j}(F), -, -, \ldots),$$

where $\sigma_{\beta,j}(F)$ is either $+$ or 0. Therefore, for each β we need no more than the pair $(j, \sigma_{\beta,j})$. Thus, let us write instead

$$\sigma(F) = (\,(k_\beta(F), \sigma_\beta(F))\,)_{\beta\in\Pi}, \tag{3}$$

where for any point λ of F, $k_\beta(F) = j$ means $j \le \langle\lambda, \beta\rangle < j + 1$, and $\sigma_\beta(F) = 0$ or $+$ according to whether $\langle\lambda, \beta\rangle$ is equal to or greater than j. We refer to (2) as the *expanded sign vector* of F and (3) as the *compact sign vector* of F.

The product of two faces of $\widetilde{\Sigma}$ is defined exactly as in the finite case. However, we can do more via the Tits cone.

2.3 The Tits cone

The *Tits cone* is a collection of polyhedral cones. We can explicitly realize this cone by embedding V in a vector space of one dimension higher and taking the cone over $\widetilde{\Sigma}$ by a point not in V. The finite Coxeter complex Σ is the boundary of the cone. That is, once linearized by taking the cone point to be the origin, all parallel hyperplanes in $\widetilde{\mathcal{H}}$ converge to a common hyperplane in the space parallel to V and containing the cone point. See Figures 2 and 3.

The faces of Σ can thus be endowed with an expanded sign vector as follows. If $F \in \Sigma$, $\sigma_{\beta,k}(F) = \sigma_\beta(F)$ for all k. The Tits cone shows us how to extend our geometric product to a product of a face $F \in \widetilde{\Sigma}$ with a face $G \in \Sigma$. In terms of expanded sign vectors we have the following.

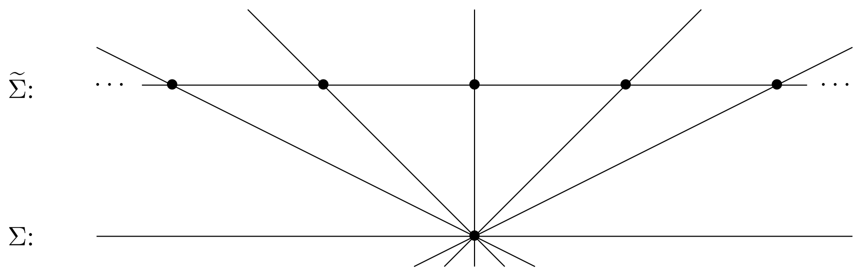

Fig. 2: The Tits cone, with $\widetilde{\Sigma}$ in the interior, Σ on the boundary.

Proposition 2.2 *Let $F \in \widetilde{\Sigma}$ and $G \in \widetilde{\Sigma} \cup \Sigma$.*

$$\sigma_{\beta,k}(FG) = \begin{cases} \sigma_{\beta,k}(F) & \text{if } \sigma_{\beta,k}(F) \neq 0, \\ \sigma_{\beta,k}(G) & \text{if } \sigma_{\beta,k}(F) = 0. \end{cases} \tag{4}$$

This product is associative, i.e., $F(G_1G_2) = (FG_1)G_2$ for all $F \in \widetilde{\Sigma}$, $G_2 \in \Sigma$, and G_1 in either $\widetilde{\Sigma}$ or Σ. The fact that the hyperplane arrangement is infinite is not a problem, since the geometric definition requires only that each face has only a finite number of faces in a small enough neighborhood. Note however that the reverse product, from $G \in \Sigma$ to $F \in \widetilde{\Sigma}$ is ill-defined. Every (full-dimensional) neighborhood of G contains infinitely many hyperplanes, so there is no "first" face to enter in walking toward F.

It is clear from this characterization that both the set $\widetilde{\mathcal{C}}$ of alcoves (maximal simplices in $\widetilde{\Sigma}$) and the set $\mathcal{C}$ of chambers take faces of $\widetilde{\Sigma}$ to alcoves, since both types of products do not leave any 0 entries in the sign vector.

We now interpret the faces of $\widetilde{\Sigma}$ in terms of $\widetilde{W}$ acting on V. The action of $\widetilde{W}$ on alcoves is simply transitive, and the fundamental alcove

$$A_\emptyset := C_\emptyset \cap \{\lambda \in V : \langle \lambda, \widetilde{\alpha} \rangle < 1\}$$

is tied to the choice of $\widetilde{S}$ in the sense that the $\widetilde{W}$-stabilizer of every point in the closure of $A_\emptyset$ (a fundamental domain) is generated by a proper subset of $\widetilde{S}$. The compact sign vector of $A_\emptyset$ is $((0,+),(0,+),\ldots))$.

We index the faces of $A_\emptyset$ by subsets of $\overline{[n]}$ so that the J-th face is

$$A_J := \begin{cases} C_J \cap \{\lambda \in V : \langle \lambda, \widetilde{\alpha} \rangle < 1\} & \text{if } 0 \notin J, \\ C_{J \setminus \{0\}} \cap \{\lambda \in V : \langle \lambda, \widetilde{\alpha} \rangle = 1\} & \text{if } 0 \in J. \end{cases}$$

Note that A_J is the empty face (or the cone point in the Tits cone) when $J = \overline{[n]}$.

Since the closure of $A_\emptyset$ is a fundamental domain for the action of $\widetilde{W}$, each face in this complex has the form $\mu + wA_J$ ($\mu \in \mathbb{Z}\Phi^\vee$, $w \in W$, $J \subseteq \overline{[n]}$). Note that the vertices of $\widetilde{\Sigma}$ are of the form $\mu + wA_{\{j\}^c}$, where $J^c := \overline{[n]} - J$. If we assign color j to each of the vertices $\mu + wA_{\{j\}^c}$, then the vertices of the cell $\mu + wA_J$ are assigned color-set J^c (without repetitions), so this coloring is balanced.

2.4 Translational invariance and the Steinberg torus

Translations are identified with 0-colored vertices, and in terms of compact sign vectors, we find

$$k_\beta(\mu + wA_J) = k_\beta(\mu) + k_\beta(wA_J) \text{ and } \sigma_\beta(\mu + wA_J) = \sigma_\beta(wA_J). \tag{5}$$

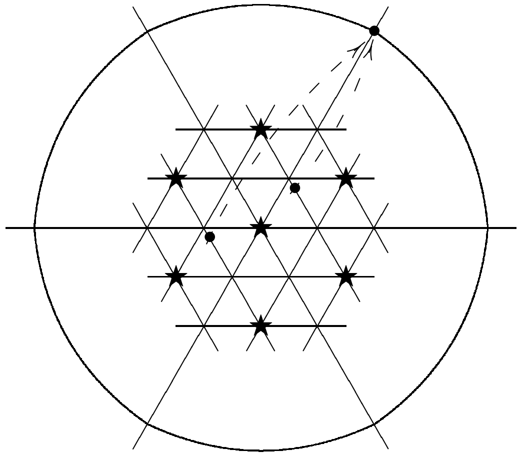

Fig. 3: The product of an affine face with a face at infinity is translation invariant. Elements of the co-root lattice $\mathbb{Z}\Phi^\vee$ are indicated with stars.

Thus for a face F we see that σ_β is completely determined by wA_J, while k_β is almost entirely controlled by μ, since $k_\beta(wA_J)$ can only be $-1, 0$, or 1.

In particular, since products with faces at infinity only change signs from 0 to +, we see that products with faces of Σ are translation invariant. See Figure 3.

Proposition 2.3 *Let $F \in \widetilde{\Sigma}$, $G \in \Sigma$, and $\mu \in \mathbb{Z}\Phi^\vee$. Then $(\mu + F)G = \mu + FG$.*

As mentioned, the product of a face $F \in \widetilde{\Sigma}$ with a chamber $C \in \Sigma$ is an alcove, $\mu + wA_\emptyset$, which we may again refer to as the *Tits projection* of F onto C. By Proposition 2.3, it suffices to characterize projections for faces wA_J, i.e., with $\mu = 0$.

Just as with faces wC_J in Σ, we find that for $i > 0$, if $\alpha_i = w^{-1}\beta$ and $(k_\beta(wA_J), \sigma_\beta(wA_J) = (0, +)$, we know i is *not* an ordinary descent of w.

If $0 \in J$, we also have

$$(k_\beta(wA_J), \sigma_\beta(wA_J)) = \begin{cases} (1, 0) & \text{if } w^{-1}\beta = \widetilde{\alpha} \\ (-1, 0) & \text{if } w^{-1}\beta = -\widetilde{\alpha}. \end{cases}$$

Thus, if $w\alpha_0 = -\beta < 0$, i.e., if 0 is an affine descent of w, then $w^{-1}\beta = -\alpha_0 = \widetilde{\alpha}$, and we get

$$0 < \langle \lambda, w^{-1}\beta \rangle = \langle \lambda, \widetilde{\alpha} \rangle \leq 1.$$

Therefore if $k_\beta(wA_J) = -1$ we know 0 is *not* an affine descent of w.

For $F \in \widetilde{\Sigma}$, let w_F denote the unique element of W such that $\mu_F + w_F A_\emptyset = FC_\emptyset$. While μ_F is uniquely determined by this projection, its exact nature does not concern us as much as w_F. Equation (4) shows that all zeroes in the expanded sign vector become $+$, and following our analysis of affine descents above allows us to generalize Proposition 2.1.

Proposition 2.4 *For any face F of $\widetilde{\Sigma}$, we have $\widetilde{D}(w_F) \subseteq \mathrm{col}(F)$. Moreover, for any $w \in W$, and any J with $\widetilde{D}(w) \subseteq J \subseteq \overline{[n]}$, there is a J-colored face F such that $w = w_F$.*

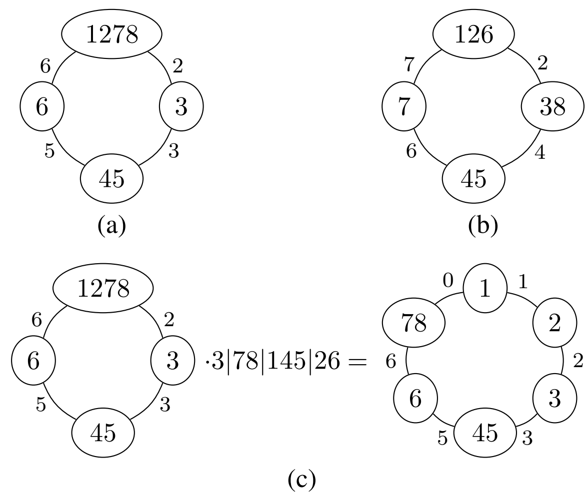

Fig. 4: (a) A spin necklace; (b) another spin necklace; (c) the product of a spin necklace and a set composition.

As the translation subgroup of $\widetilde{W}$, the co-root lattice $\mathbb{Z}\Phi^\vee$ acts as a group of color-preserving automorphisms of $\widetilde{\Sigma}$. Letting T denote the n-torus $V/\mathbb{Z}\Phi^\vee$, it follows that the image of $\widetilde{\Sigma}$ under the projection $\pi : V \to T$ is a balanced Boolean complex,

$$\overline{\Sigma} := \widetilde{\Sigma}/\mathbb{Z}\Phi^\vee.$$

Following [10], we refer to $\overline{\Sigma}$ as the *Steinberg torus*.

An alternative construction of the Steinberg torus is given by identifying maximal opposite faces of the W-invariant convex polytope

$$P_\Phi = \{\lambda \in V : -1 \le \langle \lambda, \beta \rangle \le 1 \text{ for all } \beta \in \Phi\}.$$

This polytope is the union of the closures of the alcoves $wA_\emptyset$ ($w \in W$). Note that there is a bijection with maximal faces: $w \leftrightarrow wA_\emptyset + \mathbb{Z}\Phi^\vee$ for each $w \in W$. Let $\overline{\mathcal{C}}$ denote the set of maximal faces of $\overline{\Sigma}$.

Since products of faces of $\widetilde{\Sigma}$ with faces of Σ are translation invariant (Proposition 2.3), we have a well-defined product FG with $F \in \overline{\Sigma}$ and $G \in \Sigma$. We remark, however, that products of two faces of $\widetilde{\Sigma}$ are *not* translation invariant, and so the projection π does not give a well-defined product of faces of $\overline{\Sigma}$.

We can describe faces of $\widetilde{\Sigma}(A_{n-1})$ and $\overline{\Sigma}(A_{n-1})$ in terms of a combinatorial model similar to set compositions, which we call *labeled spin necklaces*. These objects encode both the color of the face F and the representative w_F a straightforward way. (For $\widetilde{\Sigma}(A_{n-1})$ there is some mild bookkeeping involving the co-root μ_F which we will not describe here.)

First, recall the affine descent set of a permutation $w \in W = S_n$ is the set of *cyclic descents*, i.e., the descent in 0 occurs when $w_n > w_1$. For example, $\widetilde{D}(78345612) = \{2, 6\}$ while $\widetilde{D}(134625) = \{0, 4\}$.

A spin necklace consists of a cyclically ordered set partition $(B_1, \ldots, B_k)$ of $[n]$, together with labeled edges $(e_1, \ldots, e_k)$, with e_i joining B_i to B_{i+1} clockwise (and indices modulo k). The labels are the elements of the color set written in increasing order and the block B_i consists of the elements between positions $e_{i-1} + 1$ and e_i in w_F (read cyclically). Note that the difference of consecutive edge labels is the size of the intermediate block: $e_i - e_{i-1} \equiv |B_i| \mod n$. If F is such that $w_F = 78345612$ and $\mathrm{col}(F) = \{2, 3, 5, 6\}$, its spin necklace is shown in Figure 4 (a).

This restriction on the edge labels, along with the fact that $\widetilde{D}(w_F) \subseteq \mathrm{col}(F)$, allows us to uniquely recover F from a given spin necklace, e.g., the necklace in Figure 4 (b) has $w_F = 26384571$ and $\mathrm{col}(F) = \{2, 4, 6, 7\}$.

The partial order on faces corresponds to refinement of spin necklaces. The product of a face F in $\widetilde{\Sigma}$ or in $\overline{\Sigma}$ with a face G in Σ is similar to the case of two faces of Σ. The only difference between taking $F \in \widetilde{\Sigma}$ versus $F \in \overline{\Sigma}$ is that in the former case μ may change if $0 \notin \mathrm{col}(F)$. We omit the details of this change, and describe only the change in the spin necklaces.

Proposition 2.5 *Let F be a face of $\widetilde{\Sigma}(A_{n-1})$ or $\overline{\Sigma}(A_{n-1})$ with spin necklace $((e_1, B_1), \ldots, (e_k, B_k))$. Let $G = C_1|\cdots|C_l$ be a face of $\Sigma(A_{n-1})$. Then the spin necklace in the product of F and G has its blocks given by all pairwise intersections of the blocks, $B_{i,j} = B_i \cap C_j$, with edge labels $e_{i,j}$ such that $e_{i,1} = e_i$ and $e_{i,j+1} = e_{i,j} + |B_{i,j}|$.*

For example, see Figure 4 (c).

3 Modules over Solomon's descent ring

Let $\mathbb{Z}\Sigma$ denote the monoid ring of Σ and consider the subring $(\mathbb{Z}\Sigma)^W$ of W-invariants. Bidigare [6] showed that the latter is anti-isomorphic to Solomon's descent ring. We follow here the proof of this fact by Brown [8, Section 9.6], and obtain counterparts for $\widetilde{\Sigma}$ and $\overline{\Sigma}$.

The product given in Equation (1) gives the set Σ the structure of a monoid. The product in (4) turns the set $\widetilde{\Sigma}$ into a right Σ-module.

From the translational invariance of Proposition 2.3, it follows that the Steinberg torus $\overline{\Sigma}$ is a quotient right Σ-module of $\widetilde{\Sigma}$. The projection $\pi : \widetilde{\Sigma} \to \overline{\Sigma}$ is thus a morphism of right Σ-modules.

The Weyl group W acts on both $\mathbb{Z}\Phi^\vee$ and $\widetilde{\Sigma}$, and these actions and the action of $\mathbb{Z}\Phi^\vee$ on $\widetilde{\Sigma}$ are related by the *semilinearity* condition

$$w \cdot (\mu + F) = w \cdot \mu + w \cdot F \tag{6}$$

for $w \in W$, $\mu \in \mathbb{Z}\Phi^\vee$, and $F \in \widetilde{\Sigma}$. It follows that W acts on $\overline{\Sigma}$ and that π is a morphism of left W-modules.

The Weyl group W also acts on the monoid Σ and we have

$$w \cdot (FG) = (w \cdot F)(w \cdot G) \tag{7}$$

for $w \in W$, G in Σ and F in either Σ, $\widetilde{\Sigma}$, or $\overline{\Sigma}$.

We linearize the sets $\widetilde{\Sigma}$ and $\overline{\Sigma}$, obtaining abelian groups $\mathbb{Z}\widetilde{\Sigma}$ and $\mathbb{Z}\overline{\Sigma}$. We emphasize that $\mathbb{Z}\widetilde{\Sigma}$ consists of *finite* linear combinations of elements of $\widetilde{W}$. For this reason, 0 is the only element of $\mathbb{Z}\widetilde{\Sigma}$ invariant under the action of $\widetilde{W}$. We consider the action of W on the groups $\mathbb{Z}\widetilde{\Sigma}$ and $\mathbb{Z}\overline{\Sigma}$, and the corresponding subgroups of W-invariant elements. It follows from (7) that $(\mathbb{Z}\widetilde{\Sigma})^W$ is a right module over the ring $(\mathbb{Z}\Sigma)^W$, and also that the map $\pi : \widetilde{\Sigma} \to \overline{\Sigma}$ restricts to a morphism of right $(\mathbb{Z}\Sigma)^W$-modules $\pi : (\mathbb{Z}\widetilde{\Sigma})^W \to (\mathbb{Z}\overline{\Sigma})^W$.

The set of chambers $\mathcal{C}$ is a two-sided ideal of the monoid Σ. The right action of Σ on $\mathcal{C}$ is trivial: $CF = C$ for every $C \in \mathcal{C}$ and $F \in \Sigma$. The product of a face of Σ (or a face of $\widetilde{\Sigma}$, or of $\overline{\Sigma}$) and a chamber of Σ is a chamber of Σ (or an alcove of $\widetilde{\Sigma}$, or a maximal face of $\overline{\Sigma}$). This gives rise to three maps

$$\mathbb{Z}\Sigma \to \mathrm{End}_{\mathbb{Z}}(\mathbb{Z}\mathcal{C}) \qquad \mathbb{Z}\widetilde{\Sigma} \to \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}\mathcal{C}, \mathbb{Z}\widetilde{\mathcal{C}}) \qquad \mathbb{Z}\overline{\Sigma} \to \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}\mathcal{C}, \mathbb{Z}\overline{\mathcal{C}}) \tag{8}$$

denoted in every case by Φ and given by $\Phi(F)(C) := FC$ (and extended by $\mathbb{Z}$-linearity).

The abelian group $\mathrm{End}_{\mathbb{Z}}(\mathbb{Z}\mathcal{C})$ is a ring under composition, while both $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}\mathcal{C}, \mathbb{Z}\widetilde{\mathcal{C}})$ and $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}\mathcal{C}, \mathbb{Z}\overline{\mathcal{C}})$ are right $\mathrm{End}_{\mathbb{Z}}(\mathbb{Z}\mathcal{C})$-modules in the same manner. Associativity for the product of Σ (or for the right action of Σ on $\widetilde{\Sigma}$, or on $\overline{\Sigma}$) translates into the fact that $\Phi(FG) = \Phi(F) \circ \Phi(G)$ for $G \in \Sigma$ and F in either Σ, $\widetilde{\Sigma}$, or $\overline{\Sigma}$. This says that the first map in (8) is a morphism of rings, while the other two maps are morphisms of right Σ-modules, where $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}\mathcal{C}, \mathbb{Z}\widetilde{\mathcal{C}})$ and $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}\mathcal{C}, \mathbb{Z}\overline{\mathcal{C}})$ are viewed as right $\mathbb{Z}\Sigma$-modules by restriction via $\Phi : \mathbb{Z}\Sigma \to \mathrm{End}_{\mathbb{Z}}(\mathbb{Z}\mathcal{C})$.

The sets $\mathcal{C}$, $\widetilde{\mathcal{C}}$, and $\overline{\mathcal{C}}$ are stable under the action of W, and hence the groups $\mathrm{End}_{\mathbb{Z}}(\mathbb{Z}\mathcal{C})$, $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}\mathcal{C}, \mathbb{Z}\widetilde{\mathcal{C}})$, and $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}\mathcal{C}, \mathbb{Z}\overline{\mathcal{C}})$ are acted upon by W from the left. The action is $(w \cdot f)(C) = w \cdot f(w^{-1} \cdot C)$ for $w \in W$, $C \in \mathcal{C}$, and f in either $\mathrm{End}_{\mathbb{Z}}(\mathbb{Z}\mathcal{C})$, $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}\mathcal{C}, \mathbb{Z}\widetilde{\mathcal{C}})$, or $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}\mathcal{C}, \mathbb{Z}\overline{\mathcal{C}})$. Equation (7) implies that $\Phi(w \cdot F) = w \cdot \Phi(F)$ for $w \in W$ and F in either Σ, $\widetilde{\Sigma}$, or $\overline{\Sigma}$. It follows that each map Φ restricts as follows:

$$(\mathbb{Z}\Sigma)^W \to \mathrm{End}_{\mathbb{Z}}(\mathbb{Z}\mathcal{C})^W, \qquad (\mathbb{Z}\widetilde{\Sigma})^W \to \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}\mathcal{C}, \mathbb{Z}\widetilde{\mathcal{C}})^W, \qquad (\mathbb{Z}\overline{\Sigma})^W \to \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}\mathcal{C}, \mathbb{Z}\overline{\mathcal{C}})^W.$$

These maps are still denoted by Φ. The first one is a morphism of rings and the other two are morphisms of right $(\mathbb{Z}\Sigma)^W$-modules.

Since the action of W on $\mathcal{C}$ is free and transitive, we have isomorphims

$$\mathrm{End}_{\mathbb{Z}}(\mathbb{Z}\mathcal{C})^W = \mathrm{End}_{\mathbb{Z}W}(\mathbb{Z}\mathcal{C}) \cong \mathbb{Z}\mathcal{C}, \quad \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}\mathcal{C}, \mathbb{Z}\widetilde{\mathcal{C}})^W = \mathrm{Hom}_{\mathbb{Z}W}(\mathbb{Z}\mathcal{C}, \mathbb{Z}\widetilde{\mathcal{C}}) \cong \mathbb{Z}\widetilde{\mathcal{C}},$$
$$\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}\mathcal{C}, \mathbb{Z}\overline{\mathcal{C}})^W = \mathrm{Hom}_{\mathbb{Z}W}(\mathbb{Z}\mathcal{C}, \mathbb{Z}\overline{\mathcal{C}}) \cong \mathbb{Z}\overline{\mathcal{C}},$$

given in every case by $f \mapsto f(C_\emptyset)$, where $C_\emptyset$ is the fundamental chamber of Σ.

We may further identify $\mathcal{C}$ with W by means of $w \leftrightarrow w \cdot C_\emptyset$, where $C_\emptyset$ is the fundamental chamber of Σ. Consider the composite isomorphism of abelian groups

$$\mathrm{End}_{\mathbb{Z}W}(\mathbb{Z}\mathcal{C}) \cong \mathbb{Z}W. \tag{9}$$

A group element $u \in W$ corresponds to the endomorphism f such that $f(C_\emptyset) = u \cdot C_\emptyset$. If another element $v \in W$ corresponds to the endomorphism g, then $(f \circ g)(C_\emptyset) = f(v \cdot C_\emptyset) = v \cdot f(C_\emptyset) = vu \cdot C_\emptyset$, so $f \circ g$ corresponds to vu. Therefore, the isomorphism of rings (9) reverses products.

Similarly, we have $\widetilde{\mathcal{C}} \cong \widetilde{W}$ and $\overline{\mathcal{C}} \cong W$ via the actions of these groups on the fundamental alcoves of these complexes. This gives rise to isomorphisms of right $\mathrm{End}_{\mathbb{Z}W}(\mathbb{Z}\mathcal{C})$-modules

$$\mathrm{Hom}_{\mathbb{Z}W}(\mathbb{Z}\mathcal{C}, \mathbb{Z}\widetilde{\mathcal{C}}) \cong \mathbb{Z}\widetilde{W} \qquad \text{and} \qquad \mathrm{Hom}_{\mathbb{Z}W}(\mathbb{Z}\mathcal{C}, \mathbb{Z}\overline{\mathcal{C}}) \cong \mathbb{Z}W$$

where now $\mathbb{Z}\widetilde{W}$ and $\mathbb{Z}W$ are first viewed as left $\mathbb{Z}W$-modules by multiplication, and then as right $\mathrm{End}_{\mathbb{Z}W}(\mathbb{Z}\mathcal{C})$-modules via the antimorphism (9).

Composing the maps Φ with the preceding isomorphisms we obtain three maps

$$(\mathbb{Z}\Sigma)^W \to \mathbb{Z}W, \qquad (\mathbb{Z}\widetilde{\Sigma})^W \to \mathbb{Z}\widetilde{W}, \qquad (\mathbb{Z}\overline{\Sigma})^W \to \mathbb{Z}W, \tag{10}$$

denoted in every case by Ψ and given by $\Psi\left(\sum_F a_F\, F\right) = \sum_F a_F\, FC_\emptyset$, where in each case $\sum_F a_F\, F$ stands for a W-invariant element of $\mathbb{Z}\Sigma$, $\mathbb{Z}\widetilde{\Sigma}$, or $\mathbb{Z}\overline{\Sigma}$.

The first map in (10) is an anti-morphism of rings and the other two are morphisms of right $(\mathbb{Z}\Sigma)^W$-modules, where $\mathbb{Z}\widetilde{W}$ and $\mathbb{Z}W$ are first viewed as left $\mathbb{Z}W$-modules by multiplication, and then as right $(\mathbb{Z}\Sigma)^W$-modules via the antimorphism $\Psi : (\mathbb{Z}\Sigma)^W \to \mathbb{Z}W$.

The actions of W on Σ, $\widetilde{\Sigma}$ and $\overline{\Sigma}$ are color-preserving. Therefore $(\mathbb{Z}\Sigma)^W$ is a free abelian group with basis

$$\sigma_J := \sum_{F\in\Sigma_J} F,$$

where J runs over the subsets of $[n]$. Similarly, $(\mathbb{Z}\widetilde{\Sigma})^W$ and $(\mathbb{Z}\overline{\Sigma})^W$ are free abelian groups with bases

$$\widetilde{\sigma}_{J,\mu} := \sum_{F\in\widetilde{\Sigma}_{J,\mu}} F \qquad \text{and} \qquad \overline{\sigma}_J := \sum_{F\in\overline{\Sigma}_J} F,$$

where for $J \subseteq \overline{[n]}$ and $\mu \in \mathbb{Z}\Phi^\vee$, we let $\widetilde{\Sigma}_{J,\mu}$ denote the set of faces in the orbit of $\mu + A_{J^c}$ and $\overline{\Sigma}_J$ denotes the set of J-colored faces of $\overline{\Sigma}$.

For each $J \subseteq [n]$, define elements $x_J \in \mathbb{Z}W$ by

$$x_J := \sum_{w\in W:D(w)\subseteq J} w.$$

Similarly, for $\mu \in \mathbb{Z}\Phi^\vee$ and $J \subseteq \overline{[n]}$, define $\widetilde{x}_{J,\mu} \in \mathbb{Z}\widetilde{W}$ and $\overline{x}_J \in \mathbb{Z}W$ by

$$\widetilde{x}_{J,\mu} := \sum_{w\in W:\,\widetilde{D}(w)\subseteq J} (w, w\cdot\mu), \qquad \text{and} \qquad \overline{x}_J := \sum_{w\in W:\,\widetilde{D}(w)\subseteq J} w.$$

As J varies, the sets $\{w \in W : D(w) = J\}$ and $\{w \in W : \widetilde{D}(w) = J\}$ are disjoint. Therefore, each set $\{x_J\}$, $\{\widetilde{x}_{J,\mu}\}$, and $\{\overline{x}_J\}$ is linearly independent.

Proposition 3.1 *The maps Ψ behave as follows:*

$$\Psi(\sigma_J) = x_J, \qquad \Psi(\widetilde{\sigma}_{J,\mu}) = \widetilde{x}_{J,\mu}, \qquad \Psi(\overline{\sigma}_J) = \overline{x}_J.$$

In particular, Ψ is injective in every case.

Defining $\mathrm{Sol}(W) = \mathrm{Span}\{x_J : J \subseteq [n]\}$, $\widetilde{\mathrm{Sol}}(W) = \mathrm{Span}\{\widetilde{x}_{J,\mu} : J \subseteq \overline{[n]}, \mu \in \mathbb{Z}\Phi^\vee\}$, and $\overline{\mathrm{Sol}}(W) = \{\overline{x}_J : J \subseteq \overline{[n]}\}$, we have our main result.

Theorem 3.2 *The map Ψ gives the followings anti-isomorphisms:*

$$(\mathbb{Z}\Sigma)^W \to \mathrm{Sol}(W), \qquad (\mathbb{Z}\widetilde{\Sigma})^W \to \widetilde{\mathrm{Sol}}(W), \qquad \textit{and} \qquad (\mathbb{Z}\overline{\Sigma})^W \to \overline{\mathrm{Sol}}(W).$$

In particular, $\widetilde{\mathrm{Sol}}(W)$ and $\overline{\mathrm{Sol}}(W)$ are left $\mathrm{Sol}(W)$-modules.

References

[1] Peter Abramenko and Kenneth S. Brown. *Buildings*, volume 248 of *Graduate Texts in Mathematics*. Springer, New York, 2008. Theory and applications.

[2] Marcelo Aguiar, Nantel Bergeron, and Kathryn Nyman. The peak algebra and the descent algebras of types B and D. *Trans. Amer. Math. Soc.*, 356(7):2781–2824, 2004.

[3] M. D. Atkinson, G. Pfeiffer, and S. J. Van Willigenburg. The p-modular descent algebras. *Algebr. Represent. Theory*, 5(1):101–113, 2002.

[4] Pierre Baumann and Christophe Hohlweg. A Solomon descent theory for the wreath products $G \wr \mathfrak{S}_n$. *Trans. Amer. Math. Soc.*, 360(3):1475–1538 (electronic), 2008.

[5] F. Bergeron, N. Bergeron, R. B. Howlett, and D. E. Taylor. A decomposition of the descent algebra of a finite Coxeter group. *J. Algebraic Combin.*, 1(1):23–44, 1992.

[6] T. Patrick Bidigare. *Hyperplane arrangement face algebras and their associated Markov chains*. PhD thesis, University of Michigan, 1997.

[7] C. Bonnafé and G. Pfeiffer. Around Solomon's descent algebras. *Algebr. Represent. Theory*, 11(6):577–602, 2008.

[8] Kenneth S. Brown. Semigroups, rings, and Markov chains. *J. Theoret. Probab.*, 13(3):871–938, 2000.

[9] Paola Cellini. A general commutative descent algebra. *J. Algebra*, 175(3):990–1014, 1995.

[10] Kevin Dilks, T. Kyle Petersen, and John R. Stembridge. Affine descents and the Steinberg torus. *Adv. in Appl. Math.*, 42(4):423–444, 2009.

[11] Jason Fulman. Affine shuffles, shuffles with cuts, the Whitehouse module, and patience sorting. *J. Algebra*, 231(2):614–639, 2000.

[12] Jason Fulman. Descent algebras, hyperplane arrangements, and shuffling cards. *Proc. Amer. Math. Soc.*, 129(4):965–973, 2001.

[13] Adriano M. Garsia and Christophe Reutenauer. A decomposition of Solomon's descent algebra. *Adv. Math.*, 77(2):189–262, 1989.

[14] James E. Humphreys. *Reflection groups and Coxeter groups*, volume 29 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1990.

[15] Andrew Mathas and Rosa C. Orellana. Cyclotomic Solomon algebras. *Adv. Math.*, 219(2):450–487, 2008.

[16] Paul Moszkowski. Généralisation d'une formule de Solomon relative à l'anneau de groupe d'un groupe de Coxeter. *C. R. Acad. Sci. Paris Sér. I Math.*, 309(8):539–541, 1989.

[17] Frédéric Patras. L'algèbre des descentes d'une bigèbre graduée. *J. Algebra*, 170(2):547–566, 1994.

[18] T. Kyle Petersen. Cyclic descents and P-partitions. *J. Algebraic Combin.*, 22(3):343–375, 2005.

[19] Franco V. Saliola. *The face semigroup algebra of a hyperplane arrangement*. ProQuest LLC, Ann Arbor, MI, 2006. Thesis (Ph.D.)–Cornell University.

[20] Franco V. Saliola. On the quiver of the descent algebra. *J. Algebra*, 320(11):3866–3894, 2008.

[21] Franco V. Saliola. The face semigroup algebra of a hyperplane arrangement. *Canad. J. Math.*, 61(4):904–929, 2009.

[22] Louis Solomon. A Mackey formula in the group ring of a Coxeter group. *J. Algebra*, 41(2):255–264, 1976.