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Schubert polynomials and k-Schur functions (Extended abstract)

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Abstract. The main purpose of this paper is to show that the multiplication of a Schubert polynomial of finite type A by a Schur function can be understood from the multiplication in the space of dual k-Schur functions. Using earlier work by the second author, we encode both problems by means of quasisymmetric functions. On the Schubert vs. Schur side, we study the r-Bruhat order given by Bergeron-Sottile, along with certain operators associated to this order. On the other side, we connect this poset with a graph on dual k-Schur functions given by studying the affine grassmannian order of Lam-Lapointe-Morse-Shimozono. Also, we define operators associated to the graph on dual k-Schur functions which are analogous to the ones given for the Schubert vs. Schur problem.

Résumé. Le but principal de cet article est de montrer que la multiplication d'un polynôme de Schubert de type fini A par une fonction de Schur peut être comprise à partir de la multiplication dans l'espace dual des fonctions k-Schur. Les travaux antérieurs par le second auteur, nous permet de coder ces deux problèmes au moyen de fonctions quasisymétriques. Du côté Schubert vs Schur, nous étudions l'ordre partiel r-Bruhat donné par Bergeron-Sottile, ainsi que certains opérateurs associés à cet ordre. Nous donnons une relation entre l'ordre r-Bruhat et le graphe de Bruhat sur les fonctions k-Schur dualles données par l'étude de l'ordre affine grassmannienne de Lam-Lapointe-Morse-Shimozono. En outre, nous définissons des opérateurs associés a ce graphe qui sont analogues à ceux donnés pour le problème Schubert vs Schur.

Keywords: Schubert polynomials, k-Schur functions, affine grassmannian, r-Bruhat order, strong order.

1 Introduction

A fundamental problem in algebraic combinatorics is to find combinatorial rules for certain properties of a given combinatorial Hopf algebra. The problem of providing a combinatorial rule for the structure constants of a particular basis is an instance of this situation. The classical example is the Littlewood-Richardson rule which describes the multiplication and comultiplication of Schur functions within the space of symmetric functions.

Providing a rule for this kind of problems is in general very hard and many such problems are still unsolved. In particular, this paper will consider: the multiplication of Schubert polynomials, and the multiplication and comultiplication of k-Schur functions.

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Schubert polynomials are known to multiply positively since their structure constants enumerate flags in suitable triple intersections of Schubert varieties. However, there is no positive combinatorial rule to construct these constants in general. Nevertheless, since Schur polynomials correspond to grassmannian varieties which are a special class of flag varieties, we have that the Littlewood-Richardson rule is a special case of this particular problem. Even if we consider a slightly larger class of Schubert polynomials, namely, multiplication of a Schubert polynomial by a Schur function, we find that for several years there was no solution for finding a positive rule for these structure constants. Fortunately, in [6] new identities were deduced, more tools were developed and the use of techniques along the way of [2, 5, 7–9] gave as a result a combinatorial rule for this problem [3], which we will refer later as Schubert vs. Schur. Also in [3], using the work of [12], we deduce, independently of [11], a combinatorial proof that the Gromov-Witten invariants are positive.

Let us turn our attention now to k-Schur functions and their duals. In [14], one definition is shown to be related to the homology of the affine grassmannian of the affine coxeter group $\tilde{A}_{k+1}$. More precisely, the k-Schur functions are shown to be the Schubert polynomials for the affine grassmannian and, as such, the structure constants of their multiplication must be positive integers. The space of k-Schur functions span a graded Hopf algebra, and its graded dual describes the cohomology of the affine grassmannian. Thus, the comultiplication structure is also given by positive integer constants. Also, the structure constants of k-Schur functions include, as a special case, the structure of the small quantum cohomology and in particular, as mentioned above, the Gromov-Witten invariants [18].

In a series of two papers we plan to give a positive rule (along the lines of [3]) for the multiplication of dual k-Schur with a Schur function and relate this to the Schubert vs Schur problem. This is done by an in-depth study of the affine strong Bruhat graph. In order to achieve this we need to adapt the tools we have in [3, 5, 7–9] and create new ones. In this paper we start our study the strong Bruhat graph restricted to affine grassmannian permutations (see [15]). Given two such permutations u, v let $K_{[u,v]}$ be the quasisymmetric function associated to them, constructed as in [5]. The coefficient $d^v_{u,\lambda}$ of a Schur function S_λ in $K_{[u,v]}$ is the same as the coefficient of the dual k-Schur $S^{*(k)}_v$ in the product $S_\lambda S^{*(k)}_u$. In this way we recover certain structure constants of the multiplication of dual k-Schur functions since when $\lambda \subseteq (c^r)$ and $c+r = k+1$ we have that $S_\lambda = S^{(k)*}_w$ for some w affine grassmannian. We also consider an explicit combinatorial embedding of the Schubert vs. Schur problem into the dual k-Schur problem. This is done by inclusion of the chains of the grassmannian-Bruhat order into the affine strong Bruhat graph. We remark that in [13], Knutson, Lam and Speyer show that the Schubert vs. Schur problem reduces geometrically to the dual k-Schur problem. Here we focus on the positive combinatorial aspect of the problems.

The paper is organized as follows. In Sections 2 and 3 we recall some background about Schubert polynomials and k-Schur functions, respectively. In Section 4 we study the affine strong Bruhat graph and introduce the main relations satisfied by saturated chains in this order. Also,we introduce the quasi-symmetric function $K_{[u,v]}$. Finally, Section 5 is dedicated to the inclusion of the chains of the grassmannian-Bruhat order.

2 Schubert Polynomials

We recall a few results from [5, 7–9]. Let $u \in \mathcal{S}_\infty := \bigcup_{n\geq 0} \mathcal{S}_n$ be an infinite permutation where all but a finite number of positive integers are fixed. Non-affine Schubert polynomials $\mathfrak{S}_u$ are indexed by such permutations [19,20]. These polynomials form a homogenuous basis of the polynomial ring $\mathbb{Z}[x_1, x_2, \ldots]$

in countably many variables. The coefficients $c^w_{u,v}$ in $\mathfrak{S}_u\mathfrak{S}_v = \sum_v c^w_{u,v}\mathfrak{S}_w$, are known to be positive. As shown in example 6.2 of [5] (see also [9]), we can encode some of the coefficients above with a quasisymmetric function as follows. Let $\ell(w)$ be the length of a permutation $w \in \mathcal{S}_\infty$. We define the r-Bruhat order $<_r$ by its covers. Given permutations $u, w \in \mathcal{S}_\infty$, we say that $u \lessdot_r w$ if $\ell(u) + 1 = \ell(w)$ and $u^{-1}w = (i, j)$, where (i, j) is a reflection with $i \leq r < j$. When $u \lessdot_r w$, we write $wu^{-1} = (a, b)$ with $a < b$ and label the cover $u \lessdot_r w$ in the r-Bruhat order with the integer b.

We enumerate chains in the r-Bruhat order according to the descents in their sequence of labels of the edges. More precisely, we use the *descent Pieri operator*

$$x.\mathbf{H}_k \;:=\; \sum_{\omega} \mathrm{end}(\omega), \tag{2.1}$$

where the sum is over all chains ω of length k in the r-Bruhat order starting at $x \in \mathcal{S}_\infty$, $\omega \;:\; x \xrightarrow{b_1} x_1 \xrightarrow{b_2} \cdots \xrightarrow{b_k} x_k \;=:\; \mathrm{end}(\omega)$, with no descents, that is $b_1 \leq b_2 \leq \cdots \leq b_k$. Let $\langle\cdot,\cdot\rangle$ be the bilinear form on $\mathbb{Z}\mathcal{S}_\infty$ induced by the Kronecker delta function on the elements of $\mathcal{S}_\infty$. Given $u \leq_r w$, let $n = \ell(w) - \ell(u)$ be the rank of the interval $[u, w]_r$ and let

$$K_{[u,w]_r} \;=\; \sum_{\alpha \models n} \langle u.\mathbf{H}_{\alpha_1} \ldots \mathbf{H}_{\alpha_k}, w\rangle M_\alpha \tag{2.2}$$

summing over all compositions $\alpha = (\alpha_1, \ldots, \alpha_k)$ of n, where M_α is the monomial quasisymmetric function indexed by α (see [1, 5]). Now, given a saturated chain ω in the interval $[u, w]_r$ with labels $b_1, b_2, \ldots, b_n$, we let $D(\omega) = (d_1, d_2, \ldots, d_s)$ denote the unique composition of n such that $b_i > b_{i+1}$ exactly in position $i \in \{d_1, d_1 + d_2, \ldots, d_1 + d_2 + \cdots + d_{s-1}\}$. The chain ω contributes to the coefficient of M_α if and only if $\alpha \leq D(\omega)$ under refinement. We thus have

$$K_{[u,w]_r} \;=\; \sum_{\omega \in [u,w]_r} F_{D(\omega)}. \tag{2.3}$$

where F_β denotes the fundamental quasisymmetric function for a composition β.

The descent Pieri operators on this labelled poset are symmetric as $\mathbf{H}_m$ models the action of the Schur polynomial $h_m(x_1, \ldots, x_r)$ on the basis of Schubert classes (indexed by $\mathcal{S}_\infty$) in the cohomology of the flag manifold $SL(n, \mathbb{C})/B$. The quasisymmetric function $K_{[u,w]_r}$ is then a symmetric function and we can expand it in terms of Schur functions S_λ.

Proposition 2.1 ([9])

$$K_{[u,w]_r} \;=\; \sum_{\lambda} c^w_{u,(\lambda,r)}\, S_\lambda \tag{2.4}$$

where $c^w_{u,(\lambda,r)}$ is the coefficient of the Schubert polynomial $\mathfrak{S}_w$ in the product $\mathfrak{S}_u \cdot S_\lambda(x_1, \ldots, x_r)$.

Geometry shows that these coefficients $c^w_{u,(\lambda,k)}$ are non-negative. To our knowledge, the work in [3] is the first combinatorial proof of this fact.

Let us recall the combinatorial analysis in [8] to study chains in the r-Bruhat order. By definition, a saturated chain in $[u, w]_r$ of the form $\omega \;:\; u = u_0 \xrightarrow{b_1} u_1 \xrightarrow{b_2} \cdots \xrightarrow{b_n} u_n = w$, is completely characterized by the sequence of transpositions $(a_1, b_1), (a_2, b_2), \ldots (a_n, b_n)$ where $(a_i, b_i)u_{i-1} = u_i$.

Let $\mathbf{u}_{ab}\colon\ \mathbb{Z}\mathcal{S}_\infty \longrightarrow \mathbb{Z}\mathcal{S}_\infty$ denote the operator such that $\mathbf{u}_{ab}u = (a\ b)u$ if $u \lessdot_r (a,b)u$, and $\mathbf{u}_{ab}u = 0$ otherwise. We have shown in [8] that these operators satisfy the following relations:

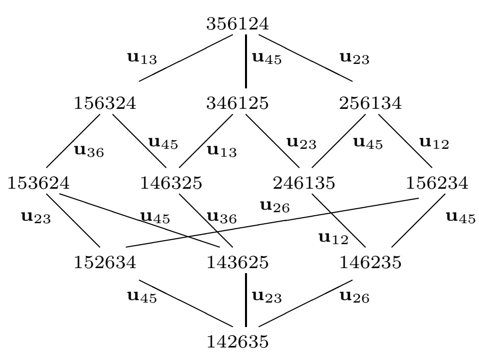
$$\begin{array}{lrcll} (1) & \mathbf{u}_{bc}\mathbf{u}_{cd}\mathbf{u}_{ac} & \equiv & \mathbf{u}_{bd}\mathbf{u}_{ab}\mathbf{u}_{bc}, & \text{if } a<b<c<d,\\ (2) & \mathbf{u}_{ac}\mathbf{u}_{cd}\mathbf{u}_{bc} & \equiv & \mathbf{u}_{bc}\mathbf{u}_{ab}\mathbf{u}_{bd}, & \text{if } a<b<c<d,\\ (3) & \mathbf{u}_{ab}\mathbf{u}_{cd} & \equiv & \mathbf{u}_{cd}\mathbf{u}_{ab}, & \text{if } b<c \text{ or } a<c<d<b,\\ (4) & \mathbf{u}_{ac}\mathbf{u}_{bd} & \equiv & \mathbf{u}_{bd}\mathbf{u}_{ac} \equiv \mathbf{0}, & \text{if } a\le b<c\le d,\\ (5) & \mathbf{u}_{bc}\mathbf{u}_{ab}\mathbf{u}_{bc} & \equiv & \mathbf{u}_{ab}\mathbf{u}_{bc}\mathbf{u}_{ab} \equiv \mathbf{0}, & \text{if } a<b<c. \end{array} \tag{2.5}$$

The $\mathbf{0}$ in relations (4) and (5) means that no chain in any r-Bruhat order can contain such a sequence of transpositions. On the other hand, relations (1), (2) and (3) are complete and transitively connect any two chains in a given interval $[u,w]_r$. It is also important to notice that the relations are independent of r. This is a fact noticed in [6]: a nonempty interval $[u,w]_r$ in the r-Bruhat order is isomorphic to a nonempty interval $[x,y]_{r'}$ in a r'-Bruhat order as long as $wu^{-1} = yx^{-1}$. This implies several identities among the structure constants.

When we write a sequence of operators $\mathbf{u}_{a_nb_n}\cdots\mathbf{u}_{a_2b_2}\mathbf{u}_{a_1b_1}$, if nonzero, it corresponds to a unique chain in some nonempty interval $[u,w]_r$ for some r and $w^{-1}u = (a_n,b_n)\cdots(a_1,b_1)$. To compute the quasisymmetric function $K_{[u,w]_r}$ as in equation (2.3), it suffices to generate one chain in $[u,w]_r$ and we can obtain the other ones using relations (1), (2) and (3) above.

Given any $\zeta \in \mathcal{S}_\infty$ we produce a chain in a nonempty interval $[u,w]_r$ as follows. Let $up(\zeta) = \{a : \zeta^{-1}(a) < a\}$. This is a finite set and we can set $r = |up(\zeta)|$. To construct w, we sort the elements in $up(\zeta) = \{i_1 < i_2 < \cdots < i_r\}$ and its complement $up^c(\zeta) = \mathbb{Z}_{>0} \setminus up(\zeta) = \{j_1 < j_2 < \ldots\}$. Next, we put $w = [i_1, i_2, \ldots, i_r, j_1, j_2, \ldots] \in \mathcal{S}_\infty$ and then we let $u = \zeta^{-1}w$. Notice that u, w and r constructed this way depend on ζ. From [6,8], we have that $[u,w]_r$ is non-empty and now we want to construct a chain in $[u,w]_r$. This is done recursively as follows: let $a_1 = u(i_1)$ where $i_1 = \max\{i \le r : u(i) < w(i)\}$ and $b_1 = u(j_1)$ where $j_1 = \min\{j > r : u(j) > u(i_1) \ge w(j)\}$ then $\mathbf{u}_{a_nb_n}\cdots\mathbf{u}_{a_2b_2}\mathbf{u}_{a_1b_1}$ is a chain in $[u,w]_r$ for any chain $\mathbf{u}_{a_nb_n}\cdots\mathbf{u}_{a_2b_2}$ in $[(a_1,b_1)u, w]_r$.

Example 2.2 Consider $\zeta = [3,6,2,5,4,1,...]$ where all other values are fixed. We have that $up(\zeta) = \{3,5,6\}$ and $up^c(\zeta) = \{1,2,4,...\}$. In this case, $r = 3$, $w = [3,5,6,1,2,4,...]$ and $u = [1,4,2,6,3,5,...]$. The recursive procedure above produces the chain $\mathbf{u}_{23}\mathbf{u}_{12}\mathbf{u}_{45}\mathbf{u}_{26}$ in $[u,v]_3$. We get all other chains by using the relations (2.5): $\mathbf{u}_{23}\mathbf{u}_{12}\mathbf{u}_{45}\mathbf{u}_{26}$, $\mathbf{u}_{23}\mathbf{u}_{12}\mathbf{u}_{26}\mathbf{u}_{45}$, $\mathbf{u}_{23}\mathbf{u}_{45}\mathbf{u}_{12}\mathbf{u}_{26}$, $\mathbf{u}_{45}\mathbf{u}_{23}\mathbf{u}_{12}\mathbf{u}_{26}$, $\mathbf{u}_{45}\mathbf{u}_{13}\mathbf{u}_{36}\mathbf{u}_{23}$, $\mathbf{u}_{13}\mathbf{u}_{45}\mathbf{u}_{36}\mathbf{u}_{23}$, $\mathbf{u}_{13}\mathbf{u}_{36}\mathbf{u}_{45}\mathbf{u}_{23}$, $\mathbf{u}_{13}\mathbf{u}_{36}\mathbf{u}_{23}\mathbf{u}_{45}$. The interval and the quasisymmetric function obtained in this case is



$$\begin{aligned} K_{[u,w]_3} &= F_{13} + 2F_{121} + 2F_{22} + F_{112}\\ &+ F_{31} + F_{211}\\ &= S_{31} + S_{22} + S_{211}. \end{aligned}$$

Notice that the functions $K_{[u,w]_r}$ encode the nonzero connected components of the given interval under the relations (2.5). In Section 5 we will show that the connected components of the chains for the r-Bruhat order where r is arbitrary, embed as a connected component of the corresponding theory for the 0-grassmannian in the affine strong Bruhat graph governing the multiplication of dual k-Schur functions.

3 k-Schur Functions and affine Grassmannians.

The k-Schur functions were originally defined combinatorially in terms of k-atoms, and conjecturally provide a positive decomposition of the Macdonald polynomials [16]. These functions have several definitions and it is conjectural that they are equivalent (see [15]). In this paper we will adopt the definition given by the k-Pieri rule and k-tableaus (see [15, 17]) since this gives us a relation with the homology and cohomology of the affine grassmannians and therefore, we get positivity in their structure constants.

The affine symmetric group W is generated by reflections s_i for $i \in \{0, 1, \ldots, k\}$, subject to the relations: $s_i^2 = 1$; $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$; $s_i s_j = s_j s_i$ if $i - j \neq \pm 1$, where $i - j$ and $i + 1$ are understood to be taken modulo $k + 1$. Let $w \in W$ and denote its length by $\ell(w)$, given by the minimal number of generators needed to write a reduced expression for w. We let W_0 denote the parabolic subgroup obtained from W by removing the generator s_0. This is naturally isomorphic to the symmetric group $\mathcal{S}_{k+1}$. For more details on affine symmetric group see [10].

Let $u \in W$ be an affine permutation. This permutation can be represented using window notation. That is, u can be seen as a bijection from $\mathbb{Z}$ to $\mathbb{Z}$, so that if u_i is the image of the integer i under u, then it can be seen as a sequence:

$$u = \cdots |u_{-k} \ \cdots \ u_{-1} \ u_0 \underbrace{|u_1 \ u_2 \ \cdots \ u_{k+1}|}_{\text{main window}} u_{k+2} \ u_{k+3} \ \cdots \ u_{2k+2}| \cdots$$

Moreover, u satisfies the property that $u_{i+k+1} = u_i + k + 1$ for all i, and the sum of the entries in the main window $u_1 + u_2 + \cdots + u_{k+1} = \binom{k+2}{2}$. Notice that in view of the first property, u is completely determined by the entries in the main window. In this notation, the generator $u = s_i$ is the permutation such that $u_{i+m(k+1)} = i + 1 + m(k+1)$ and $u_{i+1+m(k+1)} = i + m(k+1)$ for all m, and $u_j = j$ for all other values. The multiplication uw of permutations u, w in W is the usual composition given by $(uw)_i = u_{w_i}$. In view of this, the parabolic subgroup W_0 corresponds to the $u \in W$ such that the numbers $\{1, 2, \ldots, k+1\}$ appear in the main window. We will put $\bar{i} = -i$ and by convention, we consider 0 to be negative.

Now, let W^0 denote the set of minimal length coset representatives of W/W_0. In this paper we take right coset representatives, although left coset representatives could be taken also. The set of permutations in W^0 are the *affine grassmannian permutations* of W, or 0-grassmannians for short.

In this paper, any k-Schur function $S_u^{(k)}$ will be indexed by some $u \in W^0$, although k-bounded partitions or $k+1$-cores could be used instead of elements in W^0. A permutation $u \in W$ is 0-grassmannian if the numbers $1, 2, \ldots, k+1$ appear from left to right in the sequence u.

3.1 k-Schur functions.

Given $u \in W$, we say that $u \lessdot_w us_i$ is a cover for the weak order if $\ell(us_i) = \ell(u) + 1$ and we label this cover by i. The weak order on W is the transitive closure of these covers. The Pieri rule for k-Schur functions is described by certain chains in the weak order of W restricted to W^0 (see [14, 15, 17]). On the other hand, this same rule is satisfied by the Schubert grassmannian for the affine symmetric group [14].

Here, we describe the Pieri rule as follows. A saturated chain ω of length m in the weak order with end point $\mathrm{end}(\omega)$, gives us a sequence of labels $(i_1, i_2, \ldots, i_m)$. We say that the sequence $(i_1, i_2, \ldots, i_m)$ is cyclically increasing if $i_1, i_2, \ldots, i_m$ lies clockwise on a clock with hours $0, 1, \ldots, k$ and $\min\big\{j : 0 \le j \le k;\ j \notin \{i_1, i_2, \ldots, i_m\}\big\}$ lies between i_m and i_1. In particular we must have $1 \le m \le k$. Now, to express the Pieri rule, we first remark that for $1 \le m \le k$, the homogeneous symmetric function h_m corresponds to the k-Schur $S^{(k)}_{v(m)}$ where $v(m)$ is a 0-grassmannian whose main window is given by $|2\ \cdots\ m\ \bar{0}\ m+1\ \cdots\ k\ k+2|$. Then,

$$S^{(k)}_u h_m := \sum_{\omega \text{ cyclically increasing}} S^{(k)}_{\mathrm{end}(\omega)}, \tag{3.1}$$

where ω has length exactly m.
Iterating equation (3.1) one can easily see that

$$h_\lambda = \sum_u \mathrm{K}_{\lambda,u} S^{(k)}_u \tag{3.2}$$

is a triangular relation [17]. One way to define k-Schur functions is to start with equation (3.1) as a rule, and define them as follows. The *k-Schur functions* are the unique symmetric funtions $S^{(k)}_u$ obtained by inverting the matrix $[\mathrm{K}_{\lambda,u}]$ from (3.2) above.

It is clear that we can define a Pieri operator like equation (2.1) using the notion of a cyclically increasing chain. Using equation (2.2), this allows us to define a function $K_{[u,w]_w}$ for any interval in the weak order of W.

Example 3.1 Let $k = 2$ and $u = |\bar{0}\ 2\ 4|$. We consider the interval $[u, w]_w$ in the weak order where $w = |\bar{3}\ 4\ 5|$. This interval is a single chain $u = |\bar{0}\ 2\ 4| \xrightarrow{1} |2\ \bar{0}\ 4| \xrightarrow{2} |2\ 4\ \bar{0}| \xrightarrow{0} |\bar{3}\ 4\ 5| = w$. In this case, we have that $\langle u.\mathbf{H}_1\mathbf{H}_1\mathbf{H}_1, w\rangle = \langle u.\mathbf{H}_2\mathbf{H}_1, w\rangle = \langle u.\mathbf{H}_1\mathbf{H}_2, w\rangle = 1$ are the only nonzero entries in (2.2) and we get $K_{[u,w]_w} = M_{111} + M_{21} + M_{12} = F_{12} + F_{21} - F_{111} = S_{21} - S_{111}$.

This small example shows some of the behavior of the (quasi)symmetric function $K_{[u,w]_w}$ for the weak order of W. In general, it is not F-positive nor Schur positive. Although, these functions contain some information about the structure constants, it is not enough to fully understand them combinatorially, in particular, these functions lack some of the properties needed to use the theory developed in [2]. These functions were first defined in [5] in terms of the M-basis, but the definition given there in terms of the F-basis is wrong. Later on, Postnikov rediscovered them in [22] with more combinatorics involved, even though their combinatorial expansion in terms of Schur functions is still open.

3.2 Dual k-Schur functions.

Let $\Lambda = \mathbb{Z}[h_1, h_2, \ldots]$ be the Hopf algebra of symmetric functions (see [21]). The space of k-Schur functions $\Lambda_{(k)}$ can be seen as a Hopf subalgebra of Λ spanned by $\mathbb{Z}[h_1, h_2, \ldots, h_k]$ where h_i is the homogeneous symmetric function of degree i. The space Λ is a self dual Hopf algebra where the Schur functions S_λ form a self dual basis under the pairing $\langle h_\lambda, m_\mu\rangle = \delta_{\lambda,\mu}$ where the m_λ denote the monomial symmetric functions. Then, we have the inclusion $\Lambda_{(k)} \hookrightarrow \Lambda$, which turns into a projection $\Lambda \twoheadrightarrow \Lambda^{(k)}$ when passing to the dual space, where $\Lambda^{(k)} = \Lambda^*_{(k)}$ is the graded dual of $\Lambda_{(k)}$. It can be checked that the kernel of this projection is the linear span of $\{m_\lambda : \lambda_1 > k\}$, hence $\Lambda^{(k)} \cong \Lambda\big/\langle m_\lambda : \lambda_1 > k\rangle$.

The graded dual basis to $S_u^{(k)}$ will be denoted here by $\mathfrak{S}_u^{(k)} = S_u^{(k)*}$ which are also known as the affine Stanley symmetric functions. The multiplication of the dual k-Schur $\mathfrak{S}_u^{(k)}$ is described in terms of the affine Bruhat graph.

4 Affine Bruhat Graph

Let $t_{a,b}$ be the transposition in W such that for all $m \in \mathbb{Z}$, permutes $a + m(k+1)$ and $b + m(k+1)$ where $b - a \leq k$. The *affine Bruhat order* is given by its covering relation. Namely, for $u \in W$, we have $u \lessdot ut_{a,b}$ is a cover in the affine Bruhat order if $\ell(ut_{a,b}) = \ell(u) + 1$.

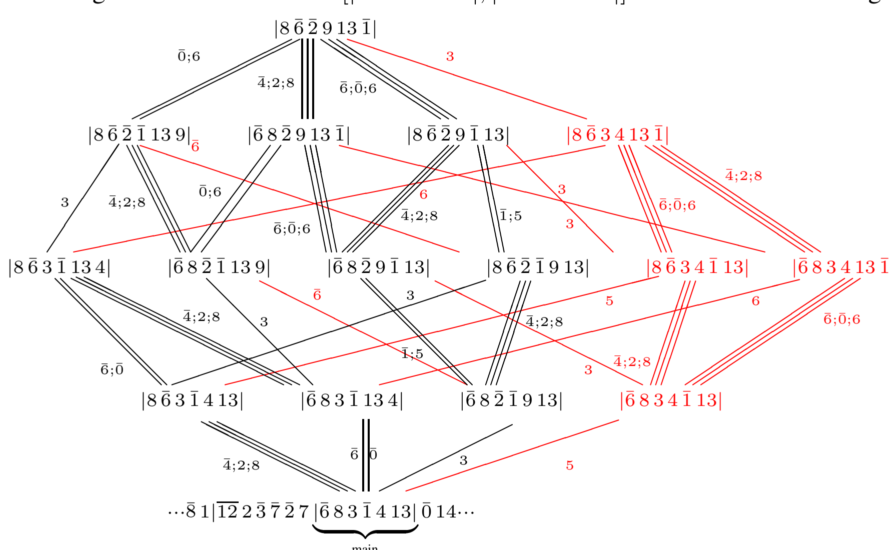
Proposition 4.1 (see [10]) *For $u \in W$ and $b - a \leq k$, we have that $u \lessdot ut_{a,b}$ is a cover in the* Bruhat order *if and only if $u(a) < u(b)$ and for all $a < i < b$ we have $u(i) < u(a)$ or $u(i) > u(b)$.*

The affine 0-Bruhat order $\lessdot_0$ arises as a suborder of the Bruhat order. For $u \in W$, a covering $u \lessdot_0 ut_{a,b}$ is encoded by transposition $t_{a,b}$ satisfying proposition 4.1 and also $u(a) \leq 0 < u(b)$. A transposition $t_{a',b'}$ satisfying the same conditions as $t_{a,b}$ gives the same affine Bruhat covering relation as long as $a' \equiv a$, $b' \equiv b$ modulo $k+1$. In view of this, we introduce a multigraph instead of a graph for the affine 0-Bruhat order, since we want to keep track of the distinct a, b such that $u \lessdot_0 ut_{a,b}$ is an affine 0-Bruhat covering for a given u.

We then define the following operators in a similar way to the ones defined in Section 2. For any $b-a \leq k+1$, let $\mathbf{t}_{ab}\colon\ \mathbb{Z}W \longrightarrow \quad \mathbb{Z}W$ be the operator on the right such that $u\mathbf{t}_{ab} = ut_{a,b}$ if $u \lessdot ut_{a,b}$ and $u(a) \leq 0 < u(b)$, and $u\mathbf{t}_{ab} = 0$ otherwise. Remark now that if $u\mathbf{t}_{ab} \neq 0$, then $u\mathbf{t}_{ab} = u\mathbf{t}_{a',b'} \neq 0$ for only finitely many values of m with $a' = a + m(k+1)$ and $b' = b + m(k+1)$.

The *affine* 0*-Bruhat graph* is the directed multigraph with vertices W and a labeled edge $u \xrightarrow{b} u\mathbf{t}_{ab}$ for every $u\mathbf{t}_{a,b} \neq 0$. We denote by $[u, w]$ the *set of paths* from u to w. Remark that all such paths will have the same length, namely $\ell(w) - \ell(u)$.

Example 4.2 We give below the interval $[|\bar{6}\,8\,3\,\bar{1}\,4\,13|, |8\,\bar{6}\,\bar{2}\,9\,13\,\bar{1}|]$ in the affine 0-Bruhat graph:



In this example we see that there are three arrows from $u = |\bar{6}\,8\,3\,\bar{1}\,4\,13|$ to $w = |8\,\bar{6}\,3\,\bar{1}\,4\,13|$, given by $u\mathbf{t}_{\bar{5}\bar{4}} = u\mathbf{t}_{12} = u\mathbf{t}_{78} = w$ and labeled $\bar{4}, 2, 8$, respectively. Also, we have that $u\mathbf{t}_{\overline{11}\,\overline{10}} = 0$. The shaded area of the graph represents the embedding of the interval in Example 2.2 as explain in the next sections.

For $u \in W^0$ such that $u\mathbf{t}_{ab} = w$, we have that $w \in W^0$ (see [15, Prop. 2.6]). In view of this nice behaviour we will restrict the affine 0-Bruhat graph to permutations in W^0.

4.1 Multiplication dual k-Schur.

For dual k-Schur functions $\mathfrak{S}_u^{(k)}$, the analogue of the Pieri formula (3.1) is given by

$$\mathfrak{S}_u^{(k)} h_m := \sum_{\substack{u\mathbf{t}_{a_1b_1}\cdots\mathbf{t}_{a_mb_m}\neq 0 \\ b_1<b_2<\ldots<b_m}} \mathfrak{S}^{(k)}_{u\mathbf{t}_{a_1b_1}\cdots\mathbf{t}_{a_mb_m}}, \tag{4.1}$$

where the sum is over all increasing paths $b_1 < b_2 < \cdots < b_m$ starting at u [15].

Since the Pieri formula is encoded by increasing chains in the affine 0-Bruhat graph restricted to W^0, we can define Pieri operators similar to equation (2.1) using increasing chains. This allows us to define the functions $K_{[u,w]}$ for any interval in the affine 0-Bruhat graph restricted to W^0. In contrast with the weak order, where we had cyclically increasing chains, any chain $\omega \in [u,w]$ has a well defined notion of descent. More precisely, for $\omega = \mathbf{t}_{a_1b_1}\mathbf{t}_{a_2b_2}\cdots\mathbf{t}_{a_mb_m}$ we have $D(\omega) = (d_1, d_2, \ldots, d_s)$ denotes the unique composition of n such that $b_i > b_{i+1}$ exactly in position $i \in \{d_1, d_1+d_2, \ldots, d_1+d_2+\cdots+d_{s-1}\}$. As in equation (2.3) we have

$$K_{[u,w]} \;=\; \sum_{\omega\in[u,v]} F_{D(\omega)} \tag{4.2}$$

and in this case $K_{[u,w]}$ is F-positive. Following [5], we have

Theorem 4.3

$$K_{[u,w]} \;=\; \sum_{\lambda} c^w_{u,\lambda}\, S_\lambda \tag{4.3}$$

where $c^w_{u,\lambda}$ is the coefficient of the dual k-Schur function $\mathfrak{S}_w^{(k)}$ in the product $\mathfrak{S}_u^{(k)} \cdot S_\lambda$.

Example 4.4 Considering the interval $[u,w] = [|\bar{6}\,8\,3\,\bar{1}\,4\,13|, |8\,\bar{6}\,\bar{2}\,9\,13\,\bar{1}|]$ we have in example 4.2. The total number of chains is 240. In this case $K_{[u,w]} = 9F_{1111} + 30F_{112} + 51F_{121} + 30F_{13} + 30F_{211} + 51F_{22} + 30F_{31} + 9F_4$, is symmetric and the expansion in term of Schur functions is positive $K_{[u,w]} = 9S_4 + 30S_{31} + 21S_{22} + 30S_{211} + 9S_{1111}$. The reader is encouraged to use SAGE and see that the coefficients are indeed the structure constants we claim in Theorem 4.3.

4.2 Relations of the operators $\mathbf{t}_{ab}$.

The purpose of this section is to understand some of the relations satisfied by the $\mathbf{t}_{ab}$ operators restricted to W^0, similar to the work done with Schubert polynomials in [3, 8]. The main theorem of this section presents the needed relations among these operators.

These relations depend on the following data. For $\mathbf{t}_{ab}$ we need to consider $a, b, \overline{a}, \overline{b}$ where $\overline{a}$ and $\overline{b}$ are the residue modulo $k+1$ of a and b respectively. Remark that $\overline{a} \neq \overline{b}$ since $b - a < k+1$. For $u \in W^0$ we have that, if non-zero, $u\mathbf{t}_{ab}$ and $u\mathbf{t}_{ab}\mathbf{t}_{cd}$ are both in W^0. The different relations satisfied by the operators $\mathbf{t}_{ab}$ and $\mathbf{t}_{cd}$ depend on the relation among $\overline{a}, \overline{b}, \overline{c}, \overline{d}$. We present some of them next.

(A) $\mathbf{t}_{ab}\mathbf{t}_{cd} \equiv \mathbf{t}_{cd}\mathbf{t}_{ab}$ if $\overline{a}, \overline{b}, \overline{c}, \overline{d}$ are distinct.

(B1) $\mathbf{t}_{ab}\mathbf{t}_{cd} \equiv \mathbf{t}_{cd}\mathbf{t}_{ab} \equiv 0$ if $(a < c < b < d)$ or $(b = c$ and $d - a > k + 1)$.

(B2) $\mathbf{t}_{ab}\mathbf{t}_{cd} \equiv 0$ if $(\overline{a} = \overline{c}$ and $b \leq d)$ or $(\overline{b} = \overline{d}$ and $c \leq a)$.

There are more possible zeros than what we present in (B), but we will satisfy ourselves with these ones for now. It will be more important to identify them in the second part of this work. Now if the numbers a, b, c, d are not distinct, then we must have $b = c$ or $d = a$. If $b = c$, then $d - a \leq k + 1$ in view of (B). Similarly if $d = a$ then $b - c \leq k + 1$.

(C1) $\mathbf{t}_{ab}\mathbf{t}_{bd} = \mathbf{t}_{ab}\mathbf{t}_{b-k-1,a}$ if $d - a = k + 1$,

(C2) $\mathbf{t}_{ab}\mathbf{t}_{bd}$ and $\mathbf{t}_{bd}\mathbf{t}_{ab}$ if $d - a < k + 1$.

Now we look at the cases $\mathbf{t}_{ab}\mathbf{t}_{cd}$ where a, b, c, d are distinct but some equalities occur between $\overline{a}, \overline{b}$ and $\overline{c}, \overline{d}$. By symmetry of the relation we will assume that $b < d$ which (excluding (B)) implies that $a < b < c < d$.

(D) $\mathbf{t}_{ab}\mathbf{t}_{cd} = \mathbf{t}_{d-k-1,c}\mathbf{t}_{b-k-1,a}$ if $\overline{b} = \overline{c},\ \overline{d} = \overline{a}$ and $(b - a) + (d - c) = k + 1$.

All the relations above are *local*. This means that if $\mathbf{t}_{ab}\mathbf{t}_{cd} = \mathbf{t}_{c'd'}\mathbf{t}_{a'b'}$, then $|a' - a|$, $|b' - b|$, $|c' - c|$ and $|d' - d|$ are strictly less than $k+1$. For example in (D) we have $|b-k-1-a|$, $|a-b|$, $|d-k-1-c|$ and $|c - d|$ which are strictly less than $k + 1$.

We now consider some more relations of length three:

(E1) $\mathbf{t}_{bc}\mathbf{t}_{cd}\mathbf{t}_{ac} \equiv \mathbf{t}_{bd}\mathbf{t}_{ab}\mathbf{t}_{bc}$ if $a < b < c < d$,

(E2) $\mathbf{t}_{ac}\mathbf{t}_{cd}\mathbf{t}_{bc} \equiv \mathbf{t}_{bc}\mathbf{t}_{ab}\mathbf{t}_{bd}$ if $a < b < c < d$.

also we have

(F) $\mathbf{t}_{bc}\mathbf{t}_{ab}\mathbf{t}_{bc} \equiv \mathbf{t}_{ab}\mathbf{t}_{bc}\mathbf{t}_{ab} \equiv \mathbf{0}$ if $a < b < c$ and $c - a < k + 1$.

Theorem 4.5 *The relations (A)–(F) above describe relations between* $\mathbf{t}$*-operators in the Strong Bruhat graph. (The proof is done case by case.)*

Remark 4.6 If we consider the permutation u we can derive more relations of length 2. Let $r = (b - a) + (d - c)$:

(X1) $u\mathbf{t}_{ab}\mathbf{t}_{cd} = u\mathbf{t}_{d,c+r}\mathbf{t}_{b-r,a}$ if $r < k+1,\ \overline{d} = \overline{a},\ u(c) \leq 0$ and $u(d) \leq 0$,

(X2) $u\mathbf{t}_{ab}\mathbf{t}_{cd} = u\mathbf{t}_{cd}\mathbf{t}_{b-r,b}$ if $r < k+1,\ \overline{d} = \overline{a}$ and $u(d) > 0$,

(X3) $u\mathbf{t}_{ab}\mathbf{t}_{cd} = u\mathbf{t}_{d-r,d}\mathbf{t}_{ab}$ if $r < k+1,\ \overline{b} = \overline{c}$ and $u(a+r) \leq 0$,

(X4) $u\mathbf{t}_{ab}\mathbf{t}_{cd} = u\mathbf{t}_{d-r,c}\mathbf{t}_{b,a+r}$ if $r < k+1,\ \overline{b} = \overline{c},\ u(b) > 0$ and $u(a+r) > 0$,

(X5) $u\mathbf{t}_{ab}\mathbf{t}_{cd} = u\mathbf{t}_{cd}\mathbf{t}_{a,b+c-d}$ if $\overline{b} = \overline{d},\ b - a > d - c$ and $u(d - b + a) > 0$,

(X6) $u\mathbf{t}_{ab}\mathbf{t}_{cd} = u\mathbf{t}_{c,d-b+a}\mathbf{t}_{a,b}$ if $\overline{b} = \overline{d},\ b - a < d - c$ and $u(a) \leq 0$.

In the (X) relations, the conditions we impose on u are minimal to assure that both sides of the equality are non-zero. These conditions are not given by the definition of the operators $\mathbf{t}_{ab}$. For example in (X1), the left hand side is non-zero regardless of the value of $u(d)$ but to guarantee that the right hand side is non-zero, we must have $u(d) \leq 0$. This shows that as operators $\mathbf{t}_{ab}\mathbf{t}_{cd} \neq \mathbf{t}_{d,c+r}\mathbf{t}_{b-r,a}$.

5 Schubert vs Schur Imbedded Inside Dual k-Schur

When comparing the relations (2.5) and the ones given in Section 4.2 we see that it may be possible to find a homomorphism from the Schubert vs Schur operators $\mathbf{u}_{ab}$ to the Dual k-Schur operators $\mathbf{t}_{a'b'}$. Such a homomorphism vanishes on many chains and this is the expected behavior.

Example 5.1 If we compare Example 2.2 and Example 4.2, the map $\mathbf{u}_{ab} \mapsto \mathbf{t}_{a-3,b-3}$ is a homomorphism that preserves all the chains from the first interval to the second one.

Now, given a non-empty interval $[x, y]_r$ in the r-Bruhat order, we want to find integers k, s and an explicit interval $[u, v]$ in the strong 0-Bruhat graph such that the homomorphism $\mathbf{u}_{ab} \mapsto \mathbf{t}_{a-s,b-s}$ maps the non-zero chains of $[x, y]_r$ to non-zero chains of $[u, v]$. In fact, we only need to assume that we have a non-zero operator $\mathbf{u}_{a_n b_n} \cdots \mathbf{u}_{a_1 b_1}$ and obtain the other ones using the corresponding relations. Then, the interval $[x, y]_r$ is isomorphic to the one described in Section 2.
For this purpose, let $\zeta = (a_n, b_n) \cdots (a_1, b_1)$, $up(\zeta) = \{i_1 < i_2 < \cdots < i_r\}$ and $up^c(\zeta) = \{j_1 < j_2 < \cdots\}$, then $r = |up(\zeta)|$. As in Section 2 we have that $[x, y]_r$ is nonempty for $y = [i_1, i_2, \ldots, i_r, j_1, j_2, \ldots]$ and $x = \zeta^{-1} y$.

Let k be such that $\alpha = x(\alpha) = y(\alpha)$ for all $\alpha > k+1$. Such a k exists since x and y have finitely many non-fixed points. Put $x_\alpha = x(\alpha)$ and take the permutation $[x_1, x_2, \ldots, x_{k+1}]$. Now, we consider the positions $\alpha_1 < \cdots < \alpha_\ell < r < \beta_1 < \cdots < \beta_t < k+1$ for which there are descents before and after r. In other words, where $x_{\alpha_i} > x_{\alpha_{i+1}}$ and $x_{\beta_j} > x_{\beta_{j+1}}$ for $1 \le i \le \ell - 1$ and $1 \le j \le t-1$. This defines segments $1, 2, \ldots, \alpha_1; \quad \cdots \quad \alpha_\ell + 1, \ldots, r; \quad r+1, \ldots, \beta_1; \quad \cdots \quad \beta_t + 1, \ldots, k+1$. We want to construct a 0-grassmannian in the $k+1$-affine permutation group W with this information such that in some adjacent $k+1$ positions we have a permutation that has the same patterns as x^{-1}. The reason we want to look at the inverse permutation x^{-1} is because the $\mathbf{u}$ operators act on the left whereas the $\mathbf{t}$ operators act on the right.
For this purpose, we first place the values $1, 2, \ldots, k+1$ on the $\mathbb{Z}$-axis as follows.

$$\begin{array}{rll}
1, 2, \ldots, k-\beta_t+1 & \text{in positions} & x_{\beta_t+1} - t(k+1), \ldots, x_{k+1} - t(k+1) \\
& \cdots & \\
k-\beta_1+2, \ldots, k-r+1 & \text{in positions} & x_{r+1}, \ldots, x_{\beta_1} \\
k-r+2, \ldots, k-\alpha_\ell+1 & \text{in positions} & x_{\alpha_\ell+1} + (k+1), \ldots, x_r + (k+1) \\
& \cdots & \\
k-\alpha_1+2, \ldots, k+1 & \text{in positions} & x_1 + (\ell+1)(k+1), \ldots, x_{\alpha_1} + (\ell+1)(k+1)
\end{array}$$

This construction places the values $1, 2, \ldots, k+1$ on the $\mathbb{Z}$-axis from left to right in distinct positions modulo $k+1$. We build a permutation u' of $\mathbb{Z}$ defining it with the relation $u'_{i+m(k+1)} = u'_i + m(k+1)$. This may not be a permutation in W as the sum $u'_1 + u'_2 + \cdots + u'_{k+1}$ may not be $\binom{k+2}{2}$, but a simple shift gives us the desired result, as shown in the next lemma (proof ommited) which will be followed by an example to make this construction clearer.

Lemma 5.2 *Any permutation u' of $\mathbb{Z}$ such that $u'_{i+m(k+1)} = u'_i + m(k+1)$ and the values $1, 2, \ldots, k+1$ are in distinct positions modulo $k+1$ satisfies $u'_1 + u'_2 + \cdots + u'_{k+1} = \binom{k+2}{2} - s(k+1)$ for some s.*

Notice that each time we shift the values of u' by 1, like $v_i = u'_{i+1}$ we get that $v_1 + v_2 + \cdots + v_{k+1} = u'_1 + u'_2 + \cdots u'_{k+1} + (k+1) = \binom{k+2}{2} + (1-s)(k+1)$. Hence, if u' is as above and if the entries

$1, 2, \ldots, k+1$ appear from left to right in u', then by defining the permutation u by $u_i = u'_{i+s}$, we get a 0-affine permutation in W^0.

Example 5.3 Let us take the permutation from Example 2.2. Let $\zeta = [3, 6, 2, 5, 4, 1, ...]$ where all other values are fixed. We can choose $k+1 = 6$. We have that $up(\zeta) = \{3, 5, 6\}$ and $up^c(\zeta) = \{1, 2, 4, ...\}$. In this case, $r = 3$, $y = [3, 5, 6, 1, 2, 4, ...]$ and $x = [1, 4, 2, 6, 3, 5, ...]$. The descents in the permutation x are in positions $\alpha = 2$ and $\beta = 4$ so that $\ell = t = 1$ and $\alpha < r < \beta$. With the procedure above, we get $1 = u'(x_5 - 6) = u'(-3),\ 2 = u'(x_6 - 6) = u'(-1);\ 3 = u'(x_4) = u'(6);\ 4 = u'(x_3 + 6) = u'(8);\ 5 = u'(x_1 + 12) = u'(13),\ 6 = u'(x_2 + 12) = u'(16)$. Once we determine the values in the positions above, all other values of u' are determined as follows

$$u' = \cdots|\overline{13}\,\bar{8}\,1\,\overline{12}\,2\,\bar{3}\,\underbrace{|\bar{7}\,\bar{2}\,7\,\bar{6}\,8\,3|}_{\text{main}}\,\bar{1}\,4\,13\,\bar{0}\,14\,9|5\,10\,19\,6\,20|\cdots$$

the sum of the entries in the main window of u' is $3 = \binom{7}{2} - 3(6)$, hence $s = 3$. We see that the entries of u' in the main window $[\bar{7}\,\bar{2}\,7\,\bar{6}\,8\,3]$ are in the same relative order as $x^{-1} = [1\,3\,5\,2\,6\,4]$. We also see that the smallest $r = 3$ entries of the main window of u' are ≤ 0 and the remaining ones are positive. Now we get u by shifting the positions of u' by s:

$$u = \cdots\overline{13}\,\bar{8}\,1|\overline{12}\,2\,\bar{3}\,\bar{7}\,\bar{2}\,7\,\underbrace{|\bar{6}\,8\,3\,\bar{1}\,4\,13|}_{\text{main}}\,\bar{0}\,14\,9\,5\,10\,19|6\,20\cdots$$

We remark that by construction, the entries $[u_{1-s}, u_{2-s}, \ldots, u_{k+1-s}]$ are the same as $[u'_1, u'_2, \ldots, u'_{k+1}]$ which in turn are in the same relative order as in x^{-1}. Therefore, from the previous paragraph we see that the smallest r entries in $[u_{1-s'}, u_{2-s'}, \ldots, u_{k+1-s'}]$ are ≤ 0 and the other entries in that window are positive. This implies that if x is covered by a non-zero permutation given by $\mathbf{u}_{ab}x$ where $x_a^{-1} \leq r < x_b^{-1}$, then we have $u\mathbf{t}_{a-s,b-s}$ is a cover in the 0-Bruhat graph. Recursively, we get that

Theorem 5.4 *Let $[x, y]_r$ be a non-empty interval $[x, y]_r$ in the r-Bruhat order and let u and s be as above. For any maximal chain $\mathbf{u}_{a_n b_n} \cdots \mathbf{u}_{a_1 b_1}$ in the interval $[x, y]_r$ we have that the chain $\mathbf{t}_{a_1-s,b_1-s} \cdots \mathbf{t}_{a_n-s,b_n-s}$ is a non-zero maximal chain in the 0-affine Bruhat graph in $[u, u\mathbf{t}_{a_1-s,b_1-s} \cdots \mathbf{t}_{a_n-s,b_n-s}]$.*

This theorem shows our main claim, namely the fact that the Schubert vs Schur problem is imbedded in the dual k-Schur problem. In the second part of our program [4] we will construct dual Knuth operators on the intervals $[u, w]$. Under the morphism above, connected components of certain dual equivalent graphs obtained in [3] are mapped to connected components of the dual equivalent graph of $[u, w]$. This shows in a stronger sense the imbedding above and explains the difficulty of the two problems. This allows us to conclude that solving the dual k-Schur problem is harder than the problem of Schubert vs Schur.

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