On the Spectra of Simplicial Rook Graphs

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Abstract. The *simplicial rook graph* $SR(d,n)$ is the graph whose vertices are the lattice points in the nth dilate of the standard simplex in $\mathbb{R}^d$, with two vertices adjacent if they differ in exactly two coordinates. We prove that the adjacency and Laplacian matrices of $SR(3,n)$ have integral spectra for every n. We conjecture that $SR(d,n)$ is integral for all d and n, and give a geometric construction of almost all eigenvectors in terms of characteristic vectors of lattice permutohedra. For $n \leq \binom{d}{2}$, we give an explicit construction of smallest-weight eigenvectors in terms of rook placements on Ferrers diagrams. The number of these eigenvectors appears to satisfy a Mahonian distribution.

Resumé. Le *graphe des tours simplicials* $SR(d,n)$ est le graphe dont les sommets sont les points du réseau dans le $n$ième dilation du simplexe standard dans $\mathbb{R}^d$; deux sommets sont adjacents s'ils différent dans exactement deux coordonnées. Nous montrons que tous les valeurs propres des matrices d'adjacence et laplacienne de $SR(3,n)$ sont entiers, pour tous les n. Nous conjecturons que les valeurs propres sont entiers pour tous d et n, et donnons une construction géometrique de presque tous les vecteurs propres en termes des vecteurs caractéristiques de permutoèdres treillis. Pour $n \leq \binom{d}{2}$, nous donnons une construction explicite des vecteurs propres de plus petits poids en termes des placements des tours sur diagrammes de Ferrers. Le nombre de ces vecteurs propres semble satisfaire une distribution Mahonian.

Keywords: simplicial rook graph, adjacency matrix, Laplacian matrix, spectral graph theory

1 Introduction

Let d and n be nonnegative integers. The *simplicial rook graph* $SR(d,n)$ is the graph with vertices

$$V(d,n) := \left\{ x = (x_1,\dots,x_d) \colon 0 \leq x_i \leq n,\ \sum_{i=1}^{d} x_i = n \right\}$$

with two vertices adjacent if they agree in all but two coordinates. This graph has $N = \binom{n+d-1}{d-1}$ vertices and is regular of degree $\delta = (d-1)n$. Geometrically, let Δ^{d-1} denote the standard simplex in $\mathbb{R}^d$ (i.e., the convex hull of the standard basis vectors $\mathbf{e}_1,\dots,\mathbf{e}_d$) and let $n\Delta^{d-1}$ denote its n^{th} dilate (i.e., the

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convex hull of $n\mathbf{e}_1, \ldots, n\mathbf{e}_d$). Then $V(d,n)$ is the set of lattice points in $n\Delta^{d-1}$, with two points adjacent if their difference is a multiple of $\mathbf{e}_i - \mathbf{e}_j$ for some i,j. Thus the independence number of $SR(d,n)$ is the maximum number of nonattacking rooks that can be placed on a simplicial chessboard with $n+1$ "squares" on each side. For $d=3$, this independence number is $\lfloor (2n+3)/3 \rfloor$ Blackburn et al. (2011); Nivasch and Lev (2005).

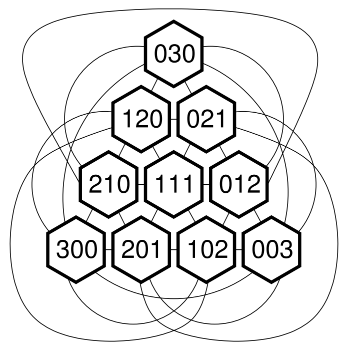

Fig. 1: The graph $SR(3,3)$.

As far as we can tell, the class of simplicial rook graphs has not been studied before. For some small values of the parameters, $SR(d,n)$ is a well-known graph: $SR(2,n)$ and $SR(d,1)$ are complete of orders $n+1$ and d respectively; $SR(3,2)$ is isomorphic to the octahedron; and $SR(d,2)$ is isomorphic to the Johnson graph $J(d+1,2)$. On the other hand, simplicial rook graphs are not in general strongly regular or distance-regular, nor are they line graphs or noncomplete extended p-sums (in the sense of (Cvetković et al., 1988, p. 55)). They are also not to be confused with the *simplicial grid graph*, in which two vertices are adjacent only if their difference vector is exactly $\mathbf{e}_i - \mathbf{e}_j$ (as opposed to some scalar multiple) nor with the *triangular graph* T_n, which is the line graph of K_n (Brouwer and Haemers, 2012, p.23), (Godsil and Royle, 2001, §10.1).

Let G be a simple graph on vertices $[n] = \{1, \ldots, n\}$. The *adjacency matrix* $A = A(G)$ is the $n \times n$ symmetric matrix whose (i,j) entry is 1 if ij is an edge, 0 otherwise. The *Laplacian matrix* is $L = L(G) = D - A$, where D is the diagonal matrix whose (i,i) entry is the degree of vertex i. The graph G is said to be *integral* (resp. *Laplacian integral*) if all eigenvalues of A (resp. L) are integers. If G is regular of degree δ, then these conditions are equivalent, since every eigenvector of A with eigenvalue λ is an eigenvector of L with eigenvalue $\delta - \lambda$.

We can now state our main theorem.

Theorem 1.1 *For every $n \geq 1$, the simplicial rook graph $SR(3,n)$ is integral and Laplacian integral, with eigenvalues as follows:*

If $n = 2m + 1$ is odd:

Eigenvalue of A	**Eigenvalue of L**	**Multiplicity**	**Eigenvector**
-3	$4m + 5 = 2n + 3$	$\binom{2m}{2}$	$\mathbf{H}_{a,b,c}$
$-2, -1, \ldots, m - 3$	$3m + 5 \ldots, 4m + 4$	3	$\mathbf{P}_k$
$m - 1$	$3m + 3$	2	$\mathbf{R}$
$m, \ldots, 2m - 1 = n - 2$	$2m + 3 \ldots, 3m + 2$	3	$\mathbf{Q}_k$
$4m + 2 = 2n$	0	1	$\mathbf{J}$

If $n = 2m$ is even:

Eigenvalue of A	**Eigenvalue of L**	**Multiplicity**	**Eigenvector**
-3	$4m + 3 = 2n + 3$	$\binom{2m-1}{2}$	$\mathbf{H}_{a,b,c}$
$-2, -1, \ldots, m - 4$	$3m + 4, \ldots, 4m + 2$	3	$\mathbf{P}_k$
$m - 3$	$3m + 3$	2	$\mathbf{R}$
$m - 1, \ldots, 2m - 2 = n - 2$	$2m + 2, \ldots, 3m + 1$	3	$\mathbf{Q}_k$
$4m = 2n$	0	1	$\mathbf{J}$

Integrality and Laplacian integrality typically arise from tightly controlled combinatorial structure in special families of graphs, including complete graphs, complete bipartite graphs and hypercubes (classical; see, e.g., (Stanley, 1999, §5.6)), Johnson graphs Krebs and Shaheen (2008), Kneser graphs Lovász (1979) and threshold graphs Merris (1994). (General references on graph eigenvalues and related topics include Balińska et al. (2002); Brouwer and Haemers (2012); Cvetković et al. (1988); Godsil and Royle (2001).) For simplicial rook graphs, lattice geometry provides this combinatorial structure. To prove Theorem 1.1, we construct a basis of $\mathbb{R}^{\binom{n+2}{2}}$ consisting of eigenvectors of $A(SR(3, n))$, as indicated in the tables above. The basis vectors $\mathbf{H}_{a,b,c}$ for the largest eigenspace are signed characteristic vectors for hexagons centered at lattice points in the interior of $n\Delta^3$ (see Figure 2). The other eigenvectors $\mathbf{P}_k, \mathbf{R}, \mathbf{Q}_k$ can be expressed as certain sums of characteristic vectors of lattice lines.

Theorem 1.1, together with Kirchhoff's matrix-tree theorem (Godsil and Royle, 2001, Lemma 13.2.4) implies the following formula for the number of spanning trees of $SR(d, n)$.

Corollary 1.2 *The number of spanning trees of $SR(3, n)$ is*

$$\begin{cases} \dfrac{32(2n+3)^{\binom{n-1}{2}} \prod_{a=n+2}^{2n+2} a^3}{3(n+1)^2(n+2)(3n+5)^3} & \textit{if } n \textit{ is odd,} \\[2ex] \dfrac{32(2n+3)^{\binom{n-1}{2}} \prod_{a=n+2}^{2n+2} a^3}{3(n+1)(n+2)^2(3n+4)^3} & \textit{if } n \textit{ is even.} \end{cases}$$

Based on experimental evidence gathered using Sage Stein et al. (2012), we make the following conjecture:

Conjecture 1.3 *The graph $SR(d, n)$ is integral for all d and n.*

We discuss the case of arbitrary d in Section 3. The construction of hexagon vectors generalizes as follows: for each permutohedron whose vertices are lattice points in $n\Delta^{d-1}$, its signed characteristic vector is an eigenvector of eigenvalue $-\binom{d}{2}$ (Proposition 3.1). This is in fact the smallest eigenvalue of $SR(d,n)$ when $n \geq \binom{d}{2}$. Moreover, these eigenvectors are linearly independent and, for fixed d, account for "almost all" of the spectrum as $n \to \infty$, in the sense that

$$\lim_{n\to\infty} \frac{\dim\,(\text{span of permutohedron eigenvectors})}{|V(d,n)|} = 1.$$

When $n < \binom{d}{2}$, the simplex $n\Delta^{d-1}$ is too small to contain any lattice permutohedra. On the other hand, the signed characteristic vectors of *partial permutohedra* (i.e., intersections of lattice permutohedra with $SR(d,n)$) are eigenvectors with eigenvalue $-n$. Experimental evidence indicates that this is in fact the smallest eigenvalue of $A(d,n)$, and that these partial permutohedra form a basis for the corresponding eigenspace. Unexpectedly, its dimension appears to be the *Mahonian number* $M(d,n)$ of permutations in $\mathfrak{S}_d$ with exactly n inversions (sequence #A008302 in Sloane (2012)). We construct a family of eigenvectors by placing rooks (ordinary rooks, not simplicial rooks!) on Ferrers boards.

The reader is referred to Martin and Wagner (2012) for the full version of this article, including proofs of all results. The authors thank Cristi Stoica for bringing their attention to references Nivasch and Lev (2005) and Blackburn et al. (2011), and Noam Elkies and several other members of the MathOverflow community for a stimulating discussion. The open-source software package Sage Stein et al. (2012) was a valuable tool in carrying out this research.

2 Proof of the Main Theorem

We begin by reviewing some basic algebraic graph theory; for a general reference, see, e.g., Godsil and Royle (2001). Let $G=(V,E)$ be a simple undirected graph with N vertices. The *adjacency matrix* $A(G)$ is the $N \times N$ matrix whose (i,j) entry is 1 if vertices i and j are adjacent, 0 otherwise. The *Laplacian matrix* is $L(G)=D(G)-A(G)$, where $D(G)$ is the diagonal matrix of vertex degrees. These are both real symmetric matrices, so they are diagonalizable, with real eigenvalues, and eigenspaces with different eigenvalues are orthogonal (Godsil and Royle, 2001, §8.4).

Proposition 2.1 *The graph $SR(d,n)$ has $N=\binom{n+d-1}{d-1}$ vertices and is regular of degree $(d-1)n$. In particular, its adjacency and Laplacian matrices have the same eigenvectors.*

Proof: Counting vertices is the classic "stars-and-bars" problem (with n stars and $d-1$ bars). For each $x \in V(d,n)$ and each pair of coordinates i,j, there are x_i+x_j other vertices that agree with x in all coordinates but i and j. Therefore, the degree of x is $\sum_{1\leq i<j\leq n}(x_i+x_j)=(d-1)\sum_{i=1}^n x_i=(d-1)n$. □

In the rest of this section, we focus exclusively on the case $d=3$, and regard n as fixed. We fix $N := \binom{n+2}{2}$, the number of vertices of $SR(3,n)$, and abbreviate $A=A(3,n)$. The matrix A acts on the vector space $\mathbb{R}^N$ with standard basis $\{\mathbf{e}_{ijk} \colon (i,j,k) \in V(3,n)\}$. We will sometimes consider the standard basis vectors as ordered lexicographically, for the purpose of showing that a collection of vectors is linearly independent.

2.1 Hexagon vectors

Let $(a,b,c) \in V(3,n)$ with $a,b,c > 0$. The corresponding *hexagon vector* is defined as

$$\mathbf{H}_{a,b,c} := \mathbf{e}_{a-1,b,c+1} - \mathbf{e}_{a,b-1,c+1} + \mathbf{e}_{a+1,b-1,c} - \mathbf{e}_{a+1,b,c-1} + \mathbf{e}_{a,b+1,c-1} - \mathbf{e}_{a-1,b+1,c}.$$

Geometrically, this is the characteristic vector, with alternating signs, of a regular lattice hexagon centered at the lattice point (a,b,c) in the interior of $n\Delta^2$ (see Figure 2).

It is not hard to check that the vectors $\mathbf{H}_{a,b,c}$ are linearly independent, and each is an eigenvector of $A(d,n)$ with eigenvalue -3. The number of possible "centers" (a,b,c) is $\binom{n-2}{3}$, so there are still $3n$ eigenvectors to determine (since $3n$ is the number of vertices of $SR(d,n)$ with at least one coordinate zero).

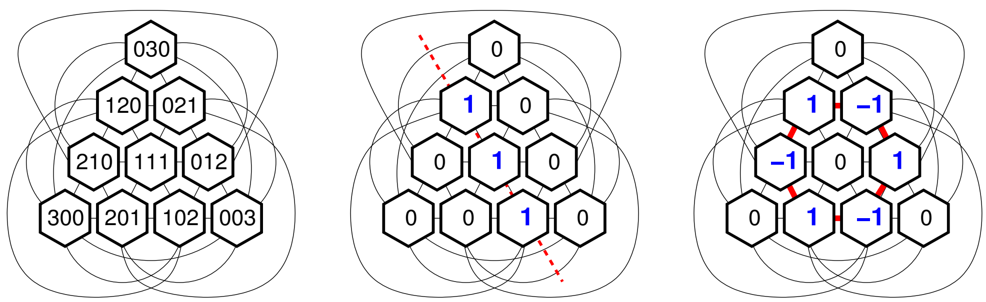

Fig. 2: (left) The graph $SR(3,3)$. (center) The vector $\mathbf{X}_1$ and the lattice line it supports. (right) $\mathbf{H}_{1,1,1}$.

Define

$$\mathbf{X}_i := \sum_{j+k=n-i} \mathbf{e}_{ijk}, \quad \mathbf{Y}_j := \sum_{i+k=n-j} \mathbf{e}_{ijk}, \quad \mathbf{Z}_k := \sum_{i+j=n-k} \mathbf{e}_{ijk}.$$

These vectors $\mathbf{X}_i, \mathbf{Y}_j, \mathbf{Z}_k$ are the characteristic vectors of lattice lines in $n\Delta^2$; see Figure 2. It can be checked that they span a vector space W of dimension $3n$, and that each one is orthogonal to every hexagon eigenvector. Therefore W is the span of all the other eigenvectors. Moreover, the symmetric group $\mathfrak{S}_3$ acts on $SR(3,n)$ (hence on each of its eigenspaces) by permuting the coordinates of vertices.

Theorem 2.2 *The eigenvectors of $A(d,n)$ are as follows.*

- *Let $\mathbf{J} = \sum_{i=0}^{n} \mathbf{X}_i = \sum_{i=0}^{n} \mathbf{Y}_i = \sum_{i=0}^{n} \mathbf{Z}_i$. Then $\mathbf{J}$ is an eigenvector with eigenvalue $2n$.*
- *Let $m = \lfloor n/2 \rfloor$ and $\mathbf{R} := \mathbf{X}_m - \mathbf{Y}_m - \mathbf{X}_{m+1} + \mathbf{Y}_{m+1}$. Then the $\mathfrak{S}_3$-orbit of $\mathbf{R}$ is an eigenspace with dimension 2 and eigenvalue $(n-6)/2$ if n is even, or $(n-3)/2$ if n is odd.*
- *For each k with $0 \le k \le \lfloor \frac{n-3}{2} \rfloor$, let*

$$\mathbf{P}_k := -(n-2k-1)(n-2k-2)\mathbf{Z}_{n-k} + \sum_{i=k+1}^{n-k-1} \Big[2(i-k-1)\mathbf{Z}_i + (2i-n)(\mathbf{X}_i + \mathbf{Y}_i) \Big].$$

Then the $\mathfrak{S}_3$-orbit of $\mathbf{P}_k$ is an eigenspace with dimension 3 and eigenvalue $k-2$.

- *For each k with $0 \le k \le \lfloor \frac{n-2}{2} \rfloor$, let*

$$\mathbf{Q}_k = (n-2k+1)(n-2k+2)\mathbf{Z}_k + \sum_{j=k}^{n-k} \Big[(2j-n)(\mathbf{X}_j + \mathbf{Y}_j) - 2(n-j-k+1)\mathbf{Z}_j\Big].$$

Then the $\mathfrak{S}_3$-orbit of $\mathbf{P}_k$ is an eigenspace with dimension 3 and eigenvalue $n-k-2$.

We omit the proof, which is a more or less direct calculation, requiring the action of $A(d,n)$ on the vectors $\mathbf{X}_i, \mathbf{Y}_j, \mathbf{Z}_k$ and several summation identities.

3 Simplicial rook graphs in arbitrary dimension

We now consider the graph $SR(d,n)$ for arbitrary d and n, with adjacency matrix $A = A(d,n)$. Recall that $SR(d,n)$ has $N := \binom{n+d-1}{d-1}$ vertices and is regular of degree $(d-1)n$. If two vertices $a = (a_1, \dots, a_d)$, $b = (b_1, \dots, b_d) \in V(d,n)$ differ only in their i^{th} and j^{th} positions (and are therefore adjacent), we write $a \underset{i,j}{\sim} b$.

Let $\mathfrak{S}_d$ be the symmetric group of order d, and let $\mathfrak{A}_d \subset \mathfrak{S}_d$ be the alternating subgroup. Let ε be the sign function

$$\varepsilon(\sigma) = \begin{cases} 1 & \text{for } \sigma \in \mathfrak{A}_d, \\ -1 & \text{for } \sigma \notin \mathfrak{A}_d. \end{cases}$$

Let $\tau_{ij} \in \mathfrak{S}_d$ denote the transposition of i and j. Note that $\mathfrak{S}_d = \mathfrak{A}_d \cup \mathfrak{A}_d \tau_{ij}$ for each i,j.

In analogy to the vectors $\mathbf{X}, \mathbf{Y}, \mathbf{Z}$ used in the $d = 3$ case, define

$$\mathbf{X}_\alpha^{(i,j)} = \mathbf{e}_\alpha + \sum_{\beta:\ \beta \underset{i,j}{\sim} \alpha} \mathbf{e}_\beta. \tag{3.1}$$

That is, $\mathbf{X}_\alpha^{(i,j)}$ is the characteristic vector of the lattice line through α in direction $\mathbf{e}_i - \mathbf{e}_j$. In particular, if $\alpha \underset{i,j}{\sim} \beta$, then $\mathbf{X}_\alpha^{(i,j)} = \mathbf{X}_\beta^{(i,j)}$. Moreover, the column of A indexed by α is

$$A\mathbf{e}_\alpha = -\binom{d}{2}\mathbf{e}_\alpha + \sum_{1 \le i < j \le d} \mathbf{X}_\alpha^{(i,j)}. \tag{3.2}$$

since e_α itself appears in each summand $\mathbf{X}_\alpha^{(i,j)}$.

3.1 Permutohedron vectors

We now generalize the construction of hexagon vectors to arbitrary dimension. The idea is that for each point p in the interior of $n\Delta^{d-1}$ and sufficiently far away from its boundary, there is a lattice permutohedron centered at p, all of whose points are vertices of $SR(d,n)$ (see Figure 3), and the signed characteristic vector of this permutohedron is an eigenvector of $A(d,n)$.

Specifically, let $w = ((1-d)/2, (3-d)/2, \dots, (d-3)/2, (d-1)/2) \in \mathbb{R}^d$. Let $p \in \mathbb{Z}^d$ (if d is odd) or $(\mathbb{Z} + \frac{1}{2})^d$ (if d is even). Then

$$\mathbf{H}_p = \sum_{\sigma \in \mathfrak{S}_d} \varepsilon(\sigma)\mathbf{e}_{p+\sigma(w)}$$

is the signed characteristic vector of the smallest lattice permutohedron with center p; we call $\mathbf{H}_p$ a *permutohedron vector*.

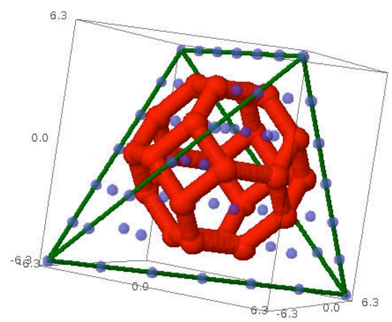

Fig. 3: A permutohedron vector ($n = 6$, $d = 4$).

Proposition 3.1 *Fix $d, n \in \mathbb{N}$, and let $p, w, \mathbf{H}_p$ be as above.*

1. *If $\{p + \sigma(w) \colon \sigma \in \mathfrak{S}_d\}$ are distinct vertices of $SR(d,n)$, then $\mathbf{H}_p$ is an eigenvector of $A(d,n)$ with eigenvalue $-\binom{d}{2}$.*
2. *The set of all such eigenvectors $\mathbf{H}_p$ is linearly independent, and its cardinality is $\binom{n-\frac{(d-1)(d-2)}{2}}{d-1}$.*

This result says that we can construct a large eigenspace by fitting many congruent permutohedra into the dilated simplex. In fact, the permutohedron eigenvectors account for "almost all" of the eigenvectors in the following sense: if $\mathcal{H}_{d,n} \subseteq \mathbb{R}^N$ is the linear span of the eigenvectors constructed in Prop. 3.1, then for each fixed d, we have

$$\lim_{n\to\infty} \frac{\dim \mathcal{H}_{d,n}}{|V(d,n)|} = \lim_{n\to\infty} \frac{\binom{n-\frac{(d-1)(d-2)}{2}}{d-1}}{\binom{n+d-1}{d-1}} = 1. \tag{3.3}$$

The characteristic vectors of lattice lines in $\mathbb{R}^d$ can be shown to be orthogonal to $\mathcal{H}_{d,n}$. We conjecture that those characteristic vectors in fact span the orthogonal complement. We have verified this statement computationally for $d = 4$ and $n \leq 11$, and for $d = 5$ and $n = 7, 8, 9$. We do not have a proof of the general statement; part of the difficulty is that it is not clear what subset of the $\mathbf{X}_\alpha^{(i,j)}$ ought to form a basis (in contrast to the case $d = 3$).

Proposition 3.2 *Suppose that $d \geq 1$ and $n \geq \binom{d}{2}$. Then the smallest eigenvalue of $SR(d,n)$ is $-\binom{d}{2}$.*

We omit the short proof, whose main idea was suggested to the authors by Noam Elkies. The smallest eigenvalue is significant in spectral graph theory; for instance, it is related to the independence number (Godsil and Royle, 2001, Lemma 9.6.2).

3.2 The small-n case and Mahonian numbers

When $n < \binom{d}{2}$, there are no permutohedron vectors — the simplex $n\Delta^{d-1}$ is too small to contain any lattice permutohedra.

Experimental evidence indicates that the smallest eigenvalue of $SR(d,n)$ is $-n$, and moreover that the multiplicity of this eigenvalue equals the number $M(d,n)$ of permutations in $\mathfrak{S}_d$ with exactly n inversions. The numbers $M(d,n)$ are well known in combinatorics as the *Mahonian numbers*, or as the coefficients of the q-factorial polynomials; see (Sloane, 2012, sequence #A008302). In the rest of this section, we construct $M(d,n)$ linearly independent eigenvectors of eigenvalue $-n$; however, we do not know how to rule out the possibility of additional eigenvectors of equal or smaller eigenvalue

We review some basics of rook theory; for a general reference, see, e.g., Butler et al. (2012). For a sequence of positive integers $c = (c_1, \ldots, c_d)$, the *skyline board* $\mathrm{Sky}(c)$ consists of a sequence of d columns, with the i^{th} column containing c_i squares. A *rook placement* on $\mathrm{Sky}(c)$ consists of a choice of one square in each column. A rook placement is *proper* if all d squares belong to different rows.

An *inversion* of a permutation $\pi = (\pi_1, \ldots, \pi_d) \in \mathfrak{S}_d$ is a pair i, j such that $i < j$ and $\pi_i > \pi_j$. Let $\mathfrak{S}_{d,n}$ denote the set of permutations of $[d]$ with exactly n inversions.

Definition 3.3 *Let $\pi \in \mathfrak{S}_{d,n}$. The* inversion word *of π is $a = a(\pi) = (a_1, \ldots, a_d)$, where*

$$a_i = \#\{j \in [d] \colon i < j \text{ and } \pi_i > \pi_j\}.$$

*Note that a is a weak composition of n with d parts, hence a vertex of $SR(d,n)$. A permutation $\sigma \in \mathfrak{S}_{d,n}$ is π-*admissible *if σ is a proper skyline rook placement on* $\mathrm{Sky}(a_1+1, \ldots, a_d+d)$*; that is, if*

$$x(\sigma) = a(\pi) + \mathbf{w} - \sigma(\mathbf{w}) = a(\pi) + \mathrm{id} - \sigma$$

is a lattice point in $n\Delta^{d-1}$. Note that the coordinates of $x(\sigma)$ sum to n, so admissibility means that its coordinates are all nonnegative. The set of all π-admissible permutations is denoted $\mathrm{Adm}(\pi)$*; that is,*

$$\mathrm{Adm}(\pi) = \{\sigma \in \mathfrak{S}_d \colon a_i - \sigma_i + i \geq 0 \quad \forall i = 1, \ldots, d\}.$$

The corresponding partial permutohedron *is*

$$\mathrm{Parp}(\pi) = \{x(\sigma) \colon \sigma \in \mathrm{Adm}(\pi)\}.$$

That is, $\mathrm{Parp}(\pi)$ *is the set of permutations corresponding to lattice points in the intersection of $n\Delta^{d-1}$ with the standard permutohedron centered at $a(\pi) + \mathbf{w}$. The* partial permutohedron vector *is the signed characteristic vector of* $\mathrm{Parp}(\pi)$*, that is,*

$$\mathbf{F}_\pi = \sum_{\sigma \in \mathrm{Parp}(\pi)} \varepsilon(\sigma) \mathbf{e}_{x(\sigma)}.$$

Example 3.4 *Let $d = 4$ and $\pi = 3142 \in \mathfrak{S}_d$. Then π has $n = 3$ inversions, namely 12, 14, 34. Its inversion word is accordingly $a = (2, 0, 1, 0)$. The π-admissible permutations are the proper skyline rook placements on $\mathrm{Sky}(2+1, 0+2, 1+3, 0+4) = \mathrm{Sky}(3, 2, 4, 4)$, namely 1234, 1243, 2134, 2143, 3124, 3142, 3214, 3241 (see Figure 4). The corresponding lattice points $x(\sigma)$ can be read off from the rook placements by counting the number of empty squares above each rook, obtaining respectively 2010, 2001, 1110, 1101, 0120, 0102, 0030, 0003; these are the neighbors of a in $\mathrm{Parp}(\pi)$. Thus $\mathbf{F}_\pi = \mathbf{e}_{2010} - \mathbf{e}_{2001} - \mathbf{e}_{1110} + \mathbf{e}_{1101} - \mathbf{e}_{0120} + \mathbf{e}_{0102} + \mathbf{e}_{0030} - \mathbf{e}_{0003}$; see Figure 5.*

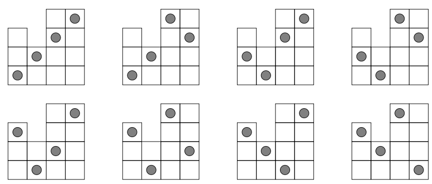

Fig. 4: Rook placements on the skyline board $\mathrm{Sky}(3, 2, 4, 4)$.

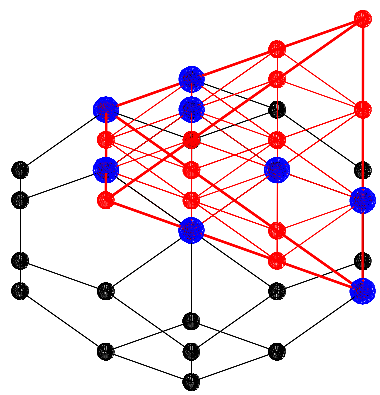

Fig. 5: The partial permutohedron $\mathrm{Parp}(3142)$ in $SR(4, 3)$.

Theorem 3.5 *Let $\pi \in \mathfrak{S}_{d,n}$ and $A = A(d, n)$. Then $\mathbf{F}_\pi$ is an eigenvector of A with eigenvalue $-n$. Moreover, for every pair d, n with $n < \binom{d}{2}$, the set $\{\mathbf{F}_\pi : \pi \in \mathfrak{S}_{d,n}\}$ is linearly independent. In particular, the dimension of the $(-n)$-eigenspace of A is at least the Mahonian number $M(d, n)$.*

Proof: We include the proof in order to illustrate the connections to (non-simplicial) rook theory. First, linear independence follows from the observation that the lexicographically leading term of $\mathbf{F}_\pi$ is $\mathbf{e}_{a(\pi)}$, and these terms are different for all $\pi \in \mathfrak{S}_{d,n}$.

Second, let $\sigma \in \mathrm{Adm}(\pi)$. Then the coefficient of $\mathbf{e}_{x(\sigma)}$ in $\mathbf{F}_\pi$ is $\varepsilon(\sigma) \in \{1, -1\}$. We will show that the coefficient of $\mathbf{e}_{x(\sigma)}$ in $A\mathbf{F}_\pi$ is $-n\varepsilon(\sigma)$, i.e., that

$$\varepsilon(\sigma)\sum_{\rho}\varepsilon(\rho) = -n, \tag{3.4}$$

the sum over all ρ such that $\rho \sim \sigma$ and $\rho \in \mathrm{Parp}(\pi)$. (Here and subsequently, $\sim$ denotes adjacency in $SR(d,n)$.) Each such rook placement ρ is obtained by multiplying σ by the transposition $(i\ j)$, that is, by choosing a rook at (i,σ_i), choosing a second rook at (j,σ_j) with $\sigma_j > \sigma_i$, and replacing these two rooks with rooks in positions (i,σ_j) and (j,σ_i). For each choice of i, there are $(a_i + i) - \sigma_i$ possible j's, and $\sum_i(a_i + i - \sigma_i) = n$. Moreover, the sign of each such ρ is opposite to that of σ, proving (3.4).

Third, let $y = (y_1,\ldots,y_d) \in V(d,n) \setminus \mathrm{Parp}(\pi)$. Then the coefficient of $e_{x(\sigma)}$ in $\mathbf{F}_\pi$ is 0. We show that the coefficient of $\mathbf{e}_{x(\sigma)}$ in $A\mathbf{F}_\pi$ is also 0, i.e., that

$$\sum_{\sigma\in N}\varepsilon(\sigma) = 0. \tag{3.5}$$

where $N = \{\rho : x(\rho) \sim y\} \cap \mathrm{Parp}(\pi)$. In order to prove this, we construct a sign-reversing involution on N. Let $a = a(\pi)$ and let $b = (b_1,\ldots,b_d) = (a_1 + 1 - y_1, a_2 + 2 - y_2, \ldots, a_d + d - y_d)$. Note that $b_i \le a_i + i$ for every i; therefore, we can regard b as a rook placement on $\mathrm{Sky}(a_1 + 1,\ldots,a_d + d)$. (It is possible that $b_i \le 0$ for one or more i; we will consider that case shortly.) To say that $y \notin \mathbf{F}_\pi$ is to say that b is not a proper π-skyline rook placement; on the other hand, we have $\sum b_i = \binom{d+1}{2}$ (as would be the case if b were proper). Hence the elements of N are the proper π-skyline rook skyline placements obtained from b by moving one rook up and one other rook down, necessarily by the same number of squares. Let $b(i \uparrow q, j \downarrow r)$ denote the rook placement obtained by moving the i^{th} rook up to row q and the j^{th} rook down to row r.

We now consider the various possible ways in which b can fail to be proper.

Case 1: $b_i \le 0$ for two or more i. In this case $N = \emptyset$, because moving only one rook up cannot produce a proper π-skyline rook placement.

Case 2: $b_i \le 0$ for exactly one i. The other rooks in b cannot all be at different heights, because that would imply that $\sum b_i \le 0 + (2 + \cdots + d) < \binom{d+1}{2}$. Therefore, either $N = \emptyset$, or else $b_j = b_k$ for some j,k and there are rooks at all heights except q and r for some $q, r < b_j = b_k$.

Then $b(i \uparrow q, j \downarrow r)$ is proper if and only if $b(i \uparrow q, k \downarrow r)$ is proper, and likewise $b(i \uparrow r, j \downarrow q)$ is proper if and only if $b(i \uparrow r, k \downarrow q)$ is proper. Each of these pairs is related by the transposition $(j\ k)$, so we have the desired sign-reversing involution on N.

Case 3: $b_i \ge 1$ for all i. Then the reason that b is not proper must be that some row has no rooks and some row has more than one rook. There are several subcases:

Case 3a: For some $q \ne r$, there are two rooks at height q, no rooks at height r, and one rook at every other height. But this is impossible because then $\sum b_i = \binom{d+1}{2} + q - r \ne \binom{d+1}{2}$.

Case 3b: There are four or more rooks at height q, or three at height q and two or more at height r. In both cases $N = \emptyset$.

Case 3c: We have $b_i = b_j = b_k$; no rooks at heights q or r for some $q < r$; and one rook at every other height. Then

$$N \subseteq \left\{ \begin{array}{lll} b(i\uparrow r, j\downarrow q), & b(j\uparrow r, i\downarrow q), & b(k\uparrow r, i\downarrow q), \\ b(i\uparrow r, k\downarrow q), & b(j\uparrow r, k\downarrow q), & b(k\uparrow r, j\downarrow q). \end{array} \right\}$$

For each column of the table above, its two rook placements are related by a transposition (e.g., $(j\ k)$ for the first column) and either both or neither of those rook placements are proper (e.g., for the first column, depending on whether or not $b_i \leq r$). Therefore, we have the desired sign-reversing involution on N.

Case 3d: We have $b_i = b_j = q$; $b_k = b_\ell = r$, and one rook at every other height except heights s and t. Now the desired sign-reversing involution on N is toggling the rook that gets moved down; for instance, $b(j \uparrow s, k \downarrow t)$ is proper if and only if $b(j \uparrow s, \ell \downarrow t)$ is proper.

This completes the proof of (3.5), which together with (3.4) completes the proof that $\mathbf{F}_\pi$ is an eigenvector of $A(d, n)$ with eigenvalue $-n$. □

Conjecture 3.6 *If $n \leq \binom{d}{2}$, then in fact $\tau(SR(d,n)) = -n$, and the dimension of the corresponding eigenspace is the Mahonian number $M(d,n)$.*

We have verified this conjecture, using Sage, for all $d \leq 6$. It is not clear in general how to rule out the possibility of a smaller eigenvalue, or of additional $(-n)$-eigenvectors linearly independent of the $\mathbf{F}_\pi$.

The proof of Theorem 3.5 implies that every partial permutohedron $\mathrm{Parp}(\pi)$ induces an n-regular subgraph of $SR(d,n)$. Another experimental observation is the following:

Conjecture 3.7 *For every $\pi \in \mathfrak{S}_{d,n}$, the induced subgraph $SR(d,n)|_{\mathrm{Parp}(\pi)}$ is Laplacian integral.*

We have verified this conjecture, using Sage, for all permutations of length $d \leq 6$. We do not know what the eigenvalues are, but these graphs are not in general strongly regular (as evidenced by the observation that they have more than 3 distinct eigenvalues).

4 Corollaries, alternate methods, and further directions

4.1 The independence number

The independence number of $SR(d,n)$ can be interpreted as the maximum number of nonattacking "rooks" that can be placed on a simplicial chessboard of side length $n+1$. By (Godsil and Royle, 2001, Lemma 9.6.2), the independence number $\alpha(G)$ of a δ-regular graph G on N vertices is at most $-\tau N/(\delta - \tau)$, where τ is the smallest eigenvalue of $A(G)$. For $d = 3$ and $n \geq 3$, we have $\tau = -3$, which implies that the independence number $\alpha(SR(d,n))$ is at most $3(n+2)(n+1)/(4n+6)$. This is of course a weaker result (except for a few small values of n) than the exact value $\lfloor (3n+3)/2 \rfloor$ obtained in Nivasch and Lev (2005) and Blackburn et al. (2011).

Question 4.1 *What is the independence number of $SR(d,n)$? That is, how many nonattacking rooks can be placed on a simplicial chessboard?*

Proposition 3.2 implies the upper bound

$$\alpha(SR(d,n)) \leq \frac{d(d+1)}{(2n+d)(d-1)} \binom{n+d-1}{d-1}$$

for $n \geq \binom{d}{2}$, but this bound is not sharp (for example, the bound for $SR(4,6)$ is $\alpha \leq 21$, but computation indicates that $\alpha = 16$).

The theory of interlacing and equitable partitions Haemers (1995), (Godsil and Royle, 2001, chapter 9) may be useful in describing the spectrum of $SR(d,n)$. Briefly, given a graph G, one constructs a square matrix P whose columns and rows correspond to orbits of vertices under the action of the automorphism group of G; under suitable conditions, every eigenvalue of P is also an eigenvalue of $A(G)$. When $G = SR(n, d)$, the spectrum of $P(G)$ appears to be a proper subset of that of $A(G)$; on the other hand, in all cases we have checked computationally ($d = 4, n \leq 30$; $d = 5, n \leq 25$), the matrices $P(SR(n, d))$ have integral spectra, which is consistent with Conjecture 1.3.

Question 4.2 *Is $SR(d, n)$ determined up to isomorphism by its spectrum?*

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