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Renormalization group-like proof of the universality of the Tutte polynomial for matroids†

G. Duchamp[1] N. Hoang-Nghia[1] T. Krajewski[2] A. Tanasa[1,3‡]

[1] *LIPN, UMR 7030 CNRS, Institut Galilée - Université Paris 13, Sorbonne Paris Cité, Villetaneuse, France*
[2] *CPT, CNRS UMR 7332, Aix Marseille Université, Campus de Luminy, Marseille, France*
[3] *Horia Hulubei National Institute for Physics and Nuclear Engineering, Magurele, Romania*

Abstract In this paper we give a new proof of the universality of the Tutte polynomial for matroids. This proof uses appropriate characters of Hopf algebra of matroids, algebra introduced by Schmitt (1994). We show that these Hopf algebra characters are solutions of some differential equations which are of the same type as the differential equations used to describe the renormalization group flow in quantum field theory. This approach allows us to also prove, in a different way, a matroid Tutte polynomial convolution formula published by Kook, Reiner and Stanton (1999). This FPSAC contribution is an extended abstract.

Résumé. Dans cet article, nous donnons une nouvelle preuve de l'universalité du polynôme de Tutte pour les matroïdes. Cette preuve utilise des caractères appropriés de l'algèbre de Hopf des matroïdes introduite par Schmitt (1994). Nous montrons que ces caractères algèbre de Hopf sont des solutions de des équations différentielles du même type que les équations différentielles utilisées pour décrire le flux du groupe de renormalisation en théorie quantique de champs. Cette approche nous permet aussi de démontrer, d'une manière différente, une formule de convolution du polynôme de Tutte des matroïdes, formule publiée par Kook, Reiner et Stanton (1999). Cette contribution FPSAC est un résumé étendu.

Keywords: Tutte polynomial for matroids, Hopf algebras for matroids, Hopf algebra characters, matroid recipe theorem, Combinatorial Physics

1 Introduction

The interplay between algebraic combinatorics and quantum field theory (QFT) has become more and more present within the spectrum of Combinatorial Physics (spectrum represented by many other subjects, such as the combinatorics of quantum mechanics, of statistical physics or of integrable systems - see, for example, Blasiak and Flajolet (2011), Blasiak et al. (2010), the review article Di Francesco (2012), the Habilitation Tanasa (2012) and references within).

One of the most known results lying at this frontier between algebraic combinatorics and QFT is the celebrated Hopf algebra Connes and Kreimer (2000), describing the combinatorics of renormalization

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in QFT. It is worth emphasizing that the coproduct of this type of Hopf algebra is based on a selection/contraction rule (one has on the coproduct left hand side (lhs) some selection of a subpart of the entity the coproduct acts on, while on the coproduct right hand side (rhs) one has the result of the *contraction* of the selected subpart). This type of rule (appearing in other situations in Mathematical Physics, see also Tanasa and Vignes-Tourneret (2008); Tanasa and Kreimer (2012); Markopoulou (2003); Tanasa (2010)) is manifestly distinct from the selection/deletion one, largely studied in algebraic combinatorics (see, for example, Duchamp et al. (2011) and references within).

In this paper, we use characters of the matroid Hopf algebra introduced in Schmitt (1994) to prove the universality property of the Tutte polynomial for matroids. We use a Combinatorial Physics approach, namely we use a renormalization group-like differential equation to prove the respective recipe theorem. Our method also allows to give a new proof of a matroid Tutte polynomial convolution formula given in Kook et al. (1999). This approach generalizes the one given in Krajewski and Martinetti (2011) for the universality of the Tutte polynomial for graphs. Moreover, the demonstrations we give here allow us to also have proofs of the graph results conjectured in Krajewski and Martinetti (2011).

The paper is structured as follows. In the following section we briefly present the renormalization group flow equation and show that equations of such type have already been successfully used in combinatorics. The third section defines matroids, as well the Tutte polynomial for matroids and the matroid Hopf algebra defined in Schmitt (1994). The following section defines some particular infinitesimal characters of this Hopf algebra as well as their exponential - proven, later on, to be non-trivially related to the Tutte polynomial for matroids. The fifth section uses all this tools to give a new proof of the matroid Tutte polynomial convolution formula given in Kook et al. (1999) and of the recipe theorem for the Tutte polynomial of matroids.

2 Renormalization group in quantum field theory - a glimpse

A QFT model (for a general introduction to QFT and not just an introduction, see for example the book Zinn-Justin (2002)) is defined by means of a functional integral of the exponential of an *action* S which, from a mathematical point of view, is a functional of the *fields* of the model. For the Φ^4 scalar model - the simplest QFT model - there is only one type of field, which we denote by $\Phi(x)$. From a mathematical point of view, for an Euclidean QFT scalar model, the field $\Phi(x)$ is a function, $\Phi : \mathbb{R}^D \to \mathbb{K}$, where D is usually taken equal to 4 (the dimension of the space) and $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ (real, respectively complex fields).

The quantities computed in QFT are generally divergent. One thus has to consider a real, positive, *cut-off* Λ - the flowing parameter. This leads to a family of cut-off dependent actions, family denoted by S_Λ. The derivation $\Lambda \frac{\partial S_\Lambda}{\partial \Lambda}$ gives the *renormalization group equation*.

The quadratic part of the action - the *propagator* of the model - can be written in the following way

$$C_{\Lambda,\Lambda_0}(p,q) = \delta(p-q) \int_{\frac{1}{\Lambda_0}}^{\frac{1}{\Lambda}} d\alpha e^{-\alpha p^2}, \tag{2.1}$$

with p and q living in the Fourier transformed space $\mathbb{R}^D$ and Λ_0 a second real, positive cut-off. In perturbative QFT, one has to consider *Feynman graphs*, and to associate to each such a graph a *Feynman integral* (further related to quantities actually measured in physical experiments). The contribution of an edge of such a Feynman graph to its associated Feynman integral is given by an integral such as (2.1).

One can then get (see Polchinski (1984)) the Polchinski flow equation

$$\Lambda \frac{\partial S_\Lambda}{\partial \Lambda} = \int_{\mathbb{R}^{2D}} \frac{1}{2} d^D p d^D q \Lambda \frac{\partial C_{\Lambda,\Lambda_0}}{\partial \Lambda} \left(\frac{\delta^2 S}{\delta \tilde{\Phi}(p) \delta \tilde{\Phi}(q)} - \frac{\delta S}{\delta \tilde{\Phi}(p)} \frac{\delta S}{\delta \tilde{\Phi}(q)} \right), \tag{2.2}$$

where $\tilde{\Phi}$ represents the Fourier transform of the function Φ. The first term in the right hand side (rhs) of the equation above corresponds to the derivation of a propagator associated to a bridge in the respective Feynman graph. The second term corresponds to an edge which is not a bridge and is part of some circuit in the graph. One can see this diagrammatically in Fig. 1.

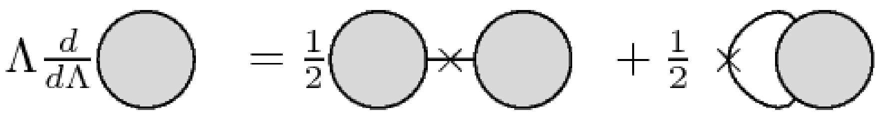

Fig. 1: Diagrammatic representation of the flow equation.

This equation can then be used to prove perturbative renormalizability in QFT. Let us also stress here, that an equation of this type is also used to prove a result of E. M. Wright which expresses the generating function of connected graphs under certain conditions (fixed excess). To get this generating functional (see, for example, Proposition $II.6$ the book Flajolet and Sedgewick (2008)), one needs to consider contributions of two types of edges (first contribution when the edge is a bridge and a second one when not - see again Fig. 1).

As already announced in the Introduction, we will use such an equation to prove the universality of the matroid Tutte polynomial (see section 5).

3 Matroids: the Tutte polynomial and the Hopf algebra

In this section we recall the definition and some properties of the Tutte polynomial for matroids as well as of the matroid Hopf algebra defined in Schmitt (1994).

Following the book Oxley (1992), one has the following definitions:

Definition 3.1 *A* **matroid** *M is a pair $(E, \mathcal{I})$ consisting of a finite set E and a collection of subsets of E satisfying:*

(I1) $\mathcal{I}$ is non-empty.

(I2) Every subset of every member of $\mathcal{I}$ is also in $\mathcal{I}$.

(I3) If X and Y are in $\mathcal{I}$ and $|X| = |Y| + 1$, then there is an element x in $X - Y$ such that $Y \cup \{x\}$ is in $\mathcal{I}$.

The set E is the ground set of the matroid and the members of $\mathcal{I}$ are the independent sets of the matroid.

One can associate matroid to graphs - graphic matroids. Nevertheless, not all matroids are graphic matroids.

Let E be an $n-$element set and let $\mathcal{I}$ be the collection of subsets of E with at most r elements, $0 \leq r \leq n$. One can check that $(E, \mathcal{I})$ is a matroid; it is called the **uniform matroid** $U_{r,n}$.

Remark 3.2 *If one takes $n = 1$, there are only two matroids, namely $U_{0,1}$ and $U_{1,1}$ and both of these matroids are graphic matroids. The graphs these two matroids correspond to are the graphs with one edge of Fig. 2 and Fig. 3. In the first case, the edge is a loop (in graph theoretical terminology) or a tadpole*

Fig. 2: The graph corresponding to the matroid $U_{0,1}$.

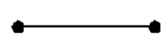

Fig. 3: The graph corresponding to the matroid $U_{1,1}$.

(in QFT language). In the second case, the edge represents a bridge (in graph theoretical terminology) or a 1-particle-reducible line (in QFT terminology) - the number of connected components of the graphs increases by 1 if one deletes the respective edge. In matroid terminology, these two particular cases correspond to a loop and respectively to a coloop (see Definitions 3.6 and respectively 3.7 below).

Definition 3.3 *The collection of maximal independent sets of a matroid are called bases. The collection of minimal dependent sets of a matroid are called circuits.*

Let $M = (E, \mathcal{I})$ be a matroid and let $\mathcal{B} = \{\mathcal{B}\}$ be the collection of bases of M. Let $\mathcal{B}^\star = \{E - B : B \in \mathcal{B}\}$. Then $\mathcal{B}^\star$ is the collection of bases of a matroid $M^\star$ on E. The matroid M^* is called the dual of M.

Definition 3.4 *Let $M = (E, \mathcal{I})$ be a matroid. The* **rank** *$r(A)$ of $A \subset E$ is defined as the cardinal of a maximal independent set in A.*

$$r(A) = max\{|B| \text{ s.t. } B \in \mathcal{I}, B \subset A\} \,. \tag{3.1}$$

Definition 3.5 *Let $M = (E, \mathcal{I})$ be a matroid with a ground set E. The* **nullity** *function is given by*

$$n(M) = |E| - r(M) \,. \tag{3.2}$$

Definition 3.6 *Let $M = (E, \mathcal{I})$ be a matroid. The element $e \in E$. is a* ***loop*** *iff $\{e\}$ is the circuit.*

Definition 3.7 *Let $M = (E, \mathcal{I})$ be a matroid. The element $e \in E$ is a* ***coloop*** *iff, for any basis B, $e \in B$.*

Let us now define two basic operations on matroids. Let M be a matroid $(E, \mathcal{I})$ and T be a subset of E. Let $\mathcal{I}' = \{I \subseteq E - T : I \in \mathcal{I}\}$. One can check that $(E - T, \mathcal{I}')$ is a matroid. We denote this matroid by $M \backslash T$ - the **deletion** of T from M. The **contraction** of T from M, M/T, is given by the formula: $M/T = (M^\star \backslash T)^\star$.

Let us also recall the following results:

Lemma 3.8 *Let M be a matroid $(E, \mathcal{I})$ and T be a subset of E. One has*

$$M|_T = M \backslash_{E-T} \,. \tag{3.3}$$

Lemma 3.9 *If e is a coloop of a matroid $M = (E, \mathcal{I})$, then $M/e = M\backslash e$.*

Lemma 3.10 *Let $M = (E, \mathcal{I})$ be a matroid and $T \subseteq E$, then, for all $X \subseteq E - T$,*

$$r_{M/T}(X) = r_M(X \cup T) - r_M(T)\ . \tag{3.4}$$

Let us now define the Tutte polynomial for matroids:

Definition 3.11 *Let $M = (E, \mathcal{I})$ be a matroid. The* **Tutte polynomial** *is given by the following formula:*

$$T_M(x, y) = \sum_{A \subseteq E} (x-1)^{r(E)-r(A)}(y-1)^{n(A)}. \tag{3.5}$$

The sum is computed over all subset of the matroid's ground set.

Definition 3.12 *Let ψ be the matroid duality map, that is a map associating to any matroid M its dual, $\psi(M) = M^\star$.*

It is worth stressing here that one can define the dual of any matroid; this is not the case for graphs, where only the dual of planar graph can be defined.

Let us recall, from Brylawsky and Oxley (1992) that

$$T_M(x, y) = T_{M^\star}(y, x). \tag{3.6}$$

In Schmitt (1994), as a particularization of a more general construction of incidence Hopf algebras, the following result was proved:

Proposition 3.13 *If $\mathcal{M}$ is a minor-closed family of matroids then $k(\widetilde{\mathcal{M}})$ is a coalgebra, with coproduct Δ and counit ϵ determined by*

$$\Delta(M) = \sum_{A \subseteq E} M|A \otimes M/A \tag{3.7}$$

and $\epsilon(M) = \begin{cases} 1, & \text{if } E = \emptyset, \\ 0 & \text{otherwise}, \end{cases}$ *for all $M = (E, \mathcal{I}) \in \mathcal{M}$. If, furthermore, the family $\mathcal{M}$ is closed under formation of direct sums, then $k(\widetilde{\mathcal{M}})$ is a Hopf algebra, with product induced by direct sum.*

We refer to this Hopf algebra as the matroid Hopf algebra. We follow Crapo and Schmitt (2005) and, by a slight abuse of notation, we denote in the same way a matroid and its isomorphic class, since the distinction will be clear from the context (as it is already in Proposition 3.13).

We denote the unit of this Hopf algebra by $\mathbf{1}$ (the empty matroid, or $U_{0,0}$).

4 Characters of matroid Hopf algebra

Let us give the following definitions:

Definition 4.1 *Let f, g be two mappings in $Hom(\mathcal{M}, \mathcal{M})$. The convolution product of f and g is given by the following formula*

$$f * g = m \circ (f \otimes g) \circ \Delta, \tag{4.1}$$

where m denotes the Hopf algebra multiplication, given here by direct sum (see above).

Definition 4.2 *A matroid Hopf algebra* **character** f *is an algebra morphism from the matroid Hopf algebra into a fixed commutative ring* $\mathbb{K}$*, such that*

$$f(M_1 \oplus M_2) = f(M_1)f(M_2), \quad f(\mathbf{1}) = 1_{\mathbb{K}}. \tag{4.2}$$

Definition 4.3 *A matroid Hopf algebra* **infinitesimal character** g *is an algebra morphism from the matroid Hopf algebra into a fixed commutative ring* $\mathbb{K}$*, such that*

$$g(M_1 \oplus M_2) = f(M_1)\epsilon(M_2) + \epsilon(M_1)g(M_2). \tag{4.3}$$

Since we work in a Hopf algebra where the non-trivial part of the coproduct is nilpotent, we can also define an exponential map by the following expression

$$\exp_*(\delta) = \epsilon + \delta + \frac{1}{2}\delta * \delta + \dots \tag{4.4}$$

where δ is an infinitesimal character.

As already stated above (see Remark 3.2), there are only two matroids with unit cardinal ground set, $U_{0,1}$ and $U_{1,1}$. We now define two maps δ_{loop} and δ_{coloop}.

$$\delta_{\mathrm{loop}}(M) = \begin{cases} 1_{\mathbb{K}} \text{ if } M = U_{0,1}, \\ 0_{\mathbb{K}} \text{ otherwise }. \end{cases} \tag{4.5}$$

$$\delta_{\mathrm{coloop}}(M) = \begin{cases} 1_{\mathbb{K}} \text{ if } M = U_{1,1}, \\ 0_{\mathbb{K}} \text{ otherwise }. \end{cases} \tag{4.6}$$

One can directly check that these maps are *infinitesimal characters* of the matroid Hopf algebra defined above.

One then has:

Lemma 4.4 *Let* $M = (E, \mathcal{I})$ *be a matroid. One has*

$$exp_*\{a\delta_{\mathrm{coloop}} + b\delta_{\mathrm{loop}}\}(M) = a^{r(M)}b^{n(M)}. \tag{4.7}$$

Proof: Using the definition (4.4), the lhs of the identity (4.7) above writes:

$$\left(\sum_{k=0}^{\infty} \frac{(a\delta_{\mathrm{coloop}} + b\delta_{\mathrm{loop}})^k}{k!}\right)(M). \tag{4.8}$$

All the terms in the sum above vanish, except the one for whom k is equal to $|E|$. Using the definition (4.1) of the convolution product, this term writes

$$\frac{1}{k!}\left(\sum_{i=0}^{k} a^{k-i}b^i \sum_{\substack{i_1,\dots,i_n \\ j_1,\dots,j_m \\ i_1+\dots+i_n=k-i \\ j_1+\dots+j_m=i}} \delta_{\mathrm{coloop}}^{\otimes(i_1)} \otimes \delta_{\mathrm{loop}}^{\otimes(j_1)} \otimes \dots \otimes \delta_{\mathrm{coloop}}^{\otimes(i_n)} \otimes \delta_{\mathrm{loop}}^{\otimes(j_m)}\right)\left(\sum_{(i)} M^{(1)} \otimes \dots \otimes M^{(k)}\right), \tag{4.9}$$

where we have used the notation $\Delta^{(k-1)}(M) = \sum_{(i)} M^{(1)} \otimes \cdots \otimes M^{(k)}$. Using the definitions (4.5) and respectively (4.6) of the infinitesimal characters δ_{loop} and respectively δ_{coloop}, implies that the submatroids $M^{(j)}$ $(j = 1, \ldots, k)$ are equal to $U_{0,1}$ or $U_{1,1}$.

Using the definition of the nullity and of the rank of a matroid concludes the proof. □

We now define the following map:

$$\alpha(x, y, s, M) := \exp_* s\{\delta_{\text{coloop}} + (y-1)\delta_{\text{loop}}\} * \exp_* s\{(x-1)\delta_{\text{coloop}} + \delta_{\text{loop}}\}(M). \tag{4.10}$$

One then has:

Proposition 4.5 *The map* (4.10) *is a character.*

Proof: The proof can be done by a direct check. On a more general basis, this is a consequence of the fact that δ_{loop} and δ_{coloop} are infinitesimal characters and the space of infinitesimal characters is a vector space; thus $s\{\delta_{\text{coloop}} + (y-1)\delta_{\text{loop}}\}$ and $s\{(x-1)\delta_{\text{coloop}} + \delta_{\text{loop}}\}$ are infinitesimal characters. Since, $\exp_*(h)$ is a character when h is an infinitesimal character and since the convolution of two characters is a character, one gets that α is a character. □

Let us define a map

$$\varphi_{a,b}(M) \longmapsto a^{r(M)} b^{n(M)} M \ . \tag{4.11}$$

Lemma 4.6 *The map $\varphi_{a,b}$ is a bialgebra automorphism.*

Proof: One can directly check that the map $\varphi_{a,b}$ is an algebra automorphism. Let us now check that this map is also a coalgebra automorphism. Using Lemma 3.8 and Lemma 3.10,

$$r(M|_T) + r(M/T) = r(M). \tag{4.12}$$

Thus, using the definitions of the map $\varphi_{a,b}$ of the matroid coproduct, one has:

$$\Delta \circ \varphi_{a,b}(M) = \sum_{T \subseteq E} (a^{r(M|_T)} b^{n(M|_T)} M|_T) \otimes (a^{r(M/_T)} b^{n(M/_T)} M/_T). \tag{4.13}$$

Using again the definition of the map $\varphi_{a,b}$ leads to

$$\Delta \circ \varphi_{a,b}(M) = (\varphi_{a,b} \otimes \varphi_{a,b}) \circ \Delta(M), \tag{4.14}$$

which concludes the proof. □

5 Proof of the universality of the Tutte polynomial for matroids

Let $M = (E, \mathcal{I})$ be a matroid. One has:

$$\begin{aligned} \alpha(x, y, s, M) \quad &= \quad \exp_* \left(s(\delta_{\text{coloop}} + (y-1)\delta_{\text{loop}}) \right) * \exp_* \left(s(-\delta_{\text{coloop}} + \delta_{\text{loop}}) \right) \\ &* \quad \exp_* \left(s(\delta_{\text{coloop}} - \delta_{\text{loop}}) \right) * \exp_* \left(s((x-1)\delta_{\text{coloop}} + \delta_{\text{loop}}) \right). \end{aligned} \tag{5.1}$$

Proposition 5.1 *Let $M = (E, \mathcal{I})$ be a matroid. The character α is related to the Tutte polynomial of matroids by the following identity:*

$$\alpha(x, y, s, M) = s^{|E|} T_M(x, y) \ . \tag{5.2}$$

Proof: Using the definition (4.1) of the convolution product in the definition (4.10) of the character α, one has the following identity:

$$\alpha(x, y, s, M) = \sum_{A \subseteq E} \exp_* s\{\delta_{\text{coloop}} + (y-1)\delta_{\text{loop}}\}(M|_A) \exp_* s\{(x-1)\delta_{\text{coloop}} + \delta_{\text{loop}}\}(M/A). \tag{5.3}$$

We can now apply Lemma 4.4 on each of the two terms in the rhs of equation (5.3) above. This leads to the result. □

Using (3.6) and the Proposition 5.1, one has:

Corollary 5.2 *One has:*

$$\alpha(x, y, s, M) = \alpha(y, x, s, M^\star). \tag{5.4}$$

This allows to give a different proof of a matroid Tutte polynomial convolution identity, which was shown in Kook et al. (1999). One has:

Corollary 5.3 *(Theorem 1 of Kook et al. (1999)) The Tutte polynomial satisfies*

$$T_M(x, y) = \sum_{A \subset E} T_{M|A}(0, y) T_{M/A}(x, 0). \tag{5.5}$$

Proof: Taking $s = 1$, this is as a direct consequence of identity (5.1), and of Proposition 5.1. □

Let us now define:

$$[f, g]_* := f * g - g * f. \tag{5.6}$$

Using the definition (4.10) of the Hopf algebra character α, one can directly prove the following result:

Proposition 5.4 *The character α is the solution of the differential equation:*

$$\frac{d\alpha}{ds} = x\alpha * \delta_{\text{coloop}} + y\delta_{\text{loop}} * \alpha + [\delta_{\text{coloop}}, \alpha]_* - [\delta_{\text{loop}}, \alpha]_* \ . \tag{5.7}$$

It is the fact that the matroid Tutte polynomial is a solution of the differential equation (5.7) that will be used now to prove the universality of the matroid Tutte polynomial. In order to do that, we take a four-variable matroid polynomial $Q_M(x, y, a, b)$ satisfying a multiplicative law and which has the following properties:

- if e is a coloop, then
$$Q_M(x, y, a, b) = xQ_{M\setminus e}(x, y, a, b) \ , \tag{5.8}$$

- if e is a loop, then
$$Q_M(x,y,a,b) = yQ_{M/e}(x,y,a,b) \tag{5.9}$$
- if e is a nonseparating point, then
$$Q_M(x,y,a,b) = aQ_{M\backslash e}(x,y,a,b) + bQ_{M/e}(x,y,a,b). \tag{5.10}$$

Remark 5.5 *Note that, when one deals with the same problem in the case of graphs, a supplementary multiplicative condition for the case of one-point joint of two graphs (i. e. identifying a vertex of the first graph and a vertex of the second graph into a single vertex of the resulting graph) is required (see, for example, Ellis-Monaghan and Merino (2010) or Sokal (2005)).*

We now define the map:
$$\beta(x,y,a,b,s,M) := s^{|E|}Q_M(x,y,a,b). \tag{5.11}$$

One then directly check (using the definition (5.11) above and the multiplicative property of the polynomial Q) that this map is again a matroid Hopf algebra character.

Proposition 5.6 *The character* (5.11) *satisfies the following differential equation:*
$$\frac{d\beta}{ds}(M) = (x\beta * \delta_{\mathrm{coloop}} + y\delta_{\mathrm{loop}} * \beta + b[\delta_{\mathrm{coloop}},\beta]_* - a[\delta_{\mathrm{loop}},\beta]_*)\,(M). \tag{5.12}$$

Proof: Applying the definition (4.1) of the convolution product, the rhs of equation (5.12) above writes
$$\begin{aligned}&= (x-b)\sum_{A\subseteq E}\beta(M|_A)\delta_{\mathrm{coloop}}(M/_A) + (y-a)\sum_{A\subseteq E}\delta_{\mathrm{loop}}(M|_A)\beta(M/_A)\\ &+ b\sum_{A\subseteq E}\delta_{\mathrm{coloop}}(M|_A)\beta(M/_A) + a\sum_{A\subseteq E}\beta(M|_A)\delta_{\mathrm{loop}}(M/_A).\end{aligned} \tag{5.13}$$

Using the definitions (4.5) and respectively (4.6) of the infinitesimal characters δ_{loop} and respectively δ_{coloop}, constraints the sums on the subsets A above. The rhs of (5.12) becomes:
$$\begin{aligned}&(x-b)\textstyle\sum_{A,M/_A=U_{1,1}}\beta(M|_A) + (y-a)\sum_{A,M|_A=U_{0,1}}\beta(M/_A)\\ &\quad+b\textstyle\sum_{A,M|_A=U_{1,1}}\beta(M/_A) + a\sum_{A,M/_A=U_{0,1}}\beta(M|_A)\end{aligned} \tag{5.14}$$

We now apply the definition of the Hopf algebra character β; one obtains:
$$\begin{aligned}&s^{|E|-1}[(x-b)\textstyle\sum_{A,M/_A=U_{1,1}}Q(x,y,a,b,M|_A) + (y-a)\sum_{A,M|_A=U_{0,1}}Q(x,y,a,b,M/_A)\\ &\quad+b\textstyle\sum_{A,M|_A=U_{1,1}}Q(x,y,a,b,M/_A) + a\sum_{A,M/_A=U_{0,1}}Q(x,y,a,b,M|_A)].\end{aligned} \tag{5.15}$$

We can now directly analyze the four particular cases $M/_A = U_{1,1}$, $M/_A = U_{0,1}$, $M|_A = U_{1,1}$ and $M|_A = U_{0,1}$:

- If $M/_A = U_{1,1}$, we can denote the ground set of $M/_A$ by $\{e\}$. Note that e is a coloop. From the Lemma 3.8, one has $M|_A = M\backslash_{E-A} = M\backslash e$. One then has $Q(x,y,a,b,M) = xQ(x,y,a,b,M|_A)$.

- If $M|_A = U_{0,1}$, then $A = \{e\}$ and e is a loop of M. Thus, one has $Q(x, y, a, b, M) = yQ(x, y, a, b, M/_A)$
- If $M|_A = U_{1,1}$, then $A = \{e\}$. One has to distinguish between two subcases:
  - e is a coloop of M. Then, by Lemma 3.9, $M/e = M\backslash e$. Thus, one has $Q(x, y, a, b, M) = xQ(x, y, a, b, M|_A)$.
  - e is a nonseparating point of M.
- If $M/_A = U_{0,1}$, one can denote the ground set of $M/_A$ by $\{e\}$. There are again two subcases to be considered:
  - e is a loop of M, one has that $M|_A = M\backslash_{(E-A)} = M\backslash_{\{e\}} = M/e$. Then one has $Q(x, y, a, b, M) = yQ(x, y, a, b, M|_A)$.
  - e is a nonseparating point of M, then one has $M|_A = M\backslash_{(E-A)} = M\backslash_{\{e\}}$

We now insert all of this in equation (5.15); this leads to three types of sums over some element e of the ground set E, e being a loop, a coloop or a nonseparating point:

$$s^{|E|-1}[\sum_{e\in E: e \text{ is a coloop}} Q(x, y, a, b, M) + \sum_{e\in E: e \text{ is a loop}} Q(x, y, a, b, M) + \sum_{e\in E: e \text{ is a regular element}} Q(x, y, a, b, M)] \tag{5.16}$$

This rewrites as

$$|E|s^{|E|-1}Q(x, y, a, b, M) = \frac{d\beta}{ds}(M), \tag{5.17}$$

which completes the proof. □

We can now state the main result of this paper, the recipe theorem specifying how to recover the matroid polynomial Q as an evaluation of the Tutte polynomial T_M:

Theorem 5.7 *One has:*

$$Q(x, y, a, b, M) = a^{n(M)}b^{r(M)}T_M(\frac{x}{b}, \frac{y}{a}). \tag{5.18}$$

Proof: The proof is a direct consequence of Propositions 5.1, 5.4 and 5.6 and of Lemma 4.6. This comes from the fact that one can apply the automorphism ϕ defined in (4.11) to the differential equation (5.12). One then obtains the differential equation (5.7) with modified parameters x/b and y/a. Finally, the solution of this differential equation is (trivially) related to the matroid Tutte polynomial T_M (see Proposition 5.1) and this concludes the proof. □

6 Conclusions

We have thus proved in this paper the universality of the matroid Tutte polynomial using differential equations of the same type as the Polchinski flow equation used in renormalization proofs in QFT. This analogy comes from the fact we differentiate with respect to two distinct type of edges of the graphs (see section 2). The role of these two types of graph edges is played in the matroid case studied here by the loop and the coloop type of elements in the matroid ground set.

As already announced in the Introduction, the matroid proofs given in this paper allow to prove the corresponding results for graphs. These graphs results were already conjectured in Krajewski and Martinetti (2011).

Let us end this paper by indicating as a possible direction for future work the investigation of the existence of a polynomial realization of matroid Hopf algebras. This objective appears as particularly interesting because polynomial realizations of Hopf algebras substantially simplify the coproduct coassociativity check (see, for example, the online version of the talk Thibon (2012)). Such an example of polynomial realizations for the Hopf algebra of trees Connes and Kreimer (1998) (amongst other combinatorial Hopf algebras) was given in Foissy et al. (2010). Let us also mention that the task of finding polynomial realizations for matroid Hopf algebras appears to us to be close to the one of finding polynomial realizations for graph Hopf algebras, since both these algebraic combinatorial structures are based on the selection/contraction rule stated in the Introduction.

References

P. Blasiak and P. Flajolet. Combinatorial models of creation-annihilation. *Séminaire Lotharingien de Combinatoire*, B65c:1–78, 2011.

P. Blasiak, G. Duchamp, A. Solomon, A. Horzela, and K. Penson. Combinatorial Algebra for second-quantized Quantum Theory. *Adv.Theor.Math.Phys.*, 14, 2010. arXiv:1001.4964.

T. Brylawsky and J. Oxley. *Matroid Applications*, volume 40 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 1992.

A. Connes and D. Kreimer. Hopf algebras, renormalization and noncommutative geometry. *Commun.Math.Phys.*, 199:203–242, 1998. doi: 10.1007/s002200050499. arXiv:hep-th/9808042.

A. Connes and D. Kreimer. Renormalization in quantum field theory and the Riemann-Hilbert problem. 1. The Hopf algebra structure of graphs and the main theorem. *Commun.Math.Phys.*, 210:249–273, 2000. doi: 10.1007/s002200050779. hep-th/9912092.

H. Crapo and W. Schmitt. A free subalgebra of the algebra of matroids. *European Journal of Combinatorics*, 26(7):1066–1085, 2005.

P. Di Francesco. Integrable Combinatorics. arXiv:1210.4514, Plenary talk given at the International Congress of Mathematical Physics, Aalborg, Denmark, August 10, 2012., 2012.

G. Duchamp, F. Hivert, J.-C. Novelli, and J.-Y. Thibon. Noncommutative symmetric functions vii: Free quasi-symmetric functions revisited. *Annals of Combinatorics*, 15:655–673, 2011.

J. Ellis-Monaghan and C. Merino. *Graph polynomial and their applications. I. The Tutte polynomial*, pages 1–42. Birkhauser, 2010. arXiv:0803.3079, invited chapter for Structural Analysis of Complex Networks.

P. Flajolet and R. Sedgewick. *Analytic combinatorics*. Cambridge University Press, 2008.

L. Foissy, J.-C. Novelli, and J.-Y. Thibon. Polynomial realizations of some combinatorial Hopf algebras. arXiv:1009.2067, 2010.

W. Kook, V. Reiner, and D. Stanton. A Convolution Formula for the Tutte Polynomial. *Journal of Combinatorial Series*, B 76:297–300, 1999.

T. Krajewski and P. Martinetti. Wilsonian renormalization, differential equations and Hopf algebras. *Contemp.Math.*, 539:187, 2011. arXiv:0806.4309.

F. Markopoulou. Coarse graining in spin foam models. *Class.Quant.Grav.*, 20:777–800, 2003. doi: 10.1088/0264-9381/20/5/301.

J. Oxley. *Matroid Theory*. Oxford University Press, New York, 1992.

J. Polchinski. Renormalization and Effective Lagragians. *Nuclear Physics*, B 231:269–299, 1984.

W. Schmitt. Incidence Hopf algebras. *Journal of Pure and Applied Algebra*, 96:299–330, 1994.

A. D. Sokal. *The multivariate Tutte polynomial (alias Potts model) for graphs and matroids*, pages 173–226. Cambridge University Press, 2005. contribution to "Surveys in Combinatorics", edited by Bridget S. Webb.

A. Tanasa. Algebraic structures in quantum gravity. *Classical and Quantum Gravity*, 27(095008), 2010.

A. Tanasa. Combinatorics in quantum field theory and random tensor models. Habilitation à diriger des recherches, Université Paris XIII, Sorbonne Paris Cité, 2012.

A. Tanasa and D. Kreimer. Combinatorial Dyson-Schwinger equations in noncommutative field theory. *Journal of Noncommutative Geometry*, (in press), 2012. arxiv:0907.2182.

A. Tanasa and F. Vignes-Tourneret. Hopf algebra of non-commutative field theory. *Journal of Noncommutative Geometry*, 2:125–139, 2008.

J.-Y. Thibon. Réalisations polynomiales d'algèbres de Hopf combinatoires. talk at the Philippe Flajolet Seminar, 27th of January 2012, available online at http://www-igm.univ-mlv.fr/ jyt/TALKS/flaj.pdf, 2012.

J. Zinn-Justin. *Quantum Field Theory and Critical Phenomena*. Oxford University Press (4th edition), 2002.