Generalized monotone triangles: an extended combinatorial reciprocity theorem

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Abstract. In a recent work, the combinatorial interpretation of the polynomial $\alpha(n; k_1, k_2, \dots, k_n)$ counting the number of Monotone Triangles with bottom row $k_1 < k_2 < \cdots < k_n$ was extended to weakly decreasing sequences $k_1 \geq k_2 \geq \cdots \geq k_n$. In this case the evaluation of the polynomial is equal to a signed enumeration of objects called Decreasing Monotone Triangles. In this paper we define Generalized Monotone Triangles – a joint generalization of both ordinary Monotone Triangles and Decreasing Monotone Triangles. As main result of the paper we prove that the evaluation of $\alpha(n; k_1, k_2, \dots, k_n)$ at arbitrary $(k_1, k_2, \dots, k_n) \in \mathbb{Z}^n$ is a signed enumeration of Generalized Monotone Triangles with bottom row $(k_1, k_2, \dots, k_n)$. Computational experiments indicate that certain evaluations of the polynomial at integral sequences yield well-known round numbers related to Alternating Sign Matrices. The main result provides a combinatorial interpretation of the conjectured identities and could turn out useful in giving bijective proofs.

Résumé. Dans un travail récent, l'interprétation combinatoire du polynôme $\alpha(n; k_1, k_2, \dots, k_n)$ comptant le nombre de triangles monotones avec dernière ligne $k_1 < k_2 < \cdots < k_n$ a été étendue aux suites faiblement décroissantes $k_1 \geq k_2 \geq \cdots \geq k_n$. Dans ce cas l'évaluation du polynôme est égale à l'énumération signée d'objets appelés triangles monotones décroissants. Dans ce papier nous définissons des triangles monotones généralisés – une généralisation commune des triangles monotones ordinaires et décroissants. Notre résultat principal est que l'évaluation de $\alpha(n; k_1, k_2, \dots, k_n)$ en un quelconque $(k_1, k_2, \dots, k_n) \in \mathbb{Z}^n$ est une énumération signée de triangles monotones généralisés avec dernière ligne $(k_1, k_2, \dots, k_n)$. Des calculs par ordinateur indiquent que certaines valeurs du polynôme sont des nombres bien connus liés aux matrices à signe alternant. Le résultat principal fournit une interprétation combinatoire des identités conjecturales et pourrait être utile dans l'obtention de preuves bijectives.

Keywords: Combinatorial Reciprocity, Monotone Triangle, Generalized Monotone Triangle, Alternating Sign Matrix

1 Introduction

A *Monotone Triangle* of size n is a triangular array of integers $(a_{i,j})_{1 \leq j \leq i \leq n}$

$$\begin{array}{ccccccc} & & & a_{1,1} & & & \\ & & a_{2,1} & & a_{2,2} & & \\ & \iddots & & & & \ddots & \\ a_{n,1} & & \cdots & & \cdots & & a_{n,n} \end{array}$$

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with strictly increasing rows, i.e. $a_{i,j} < a_{i,j+1}$, and weakly increasing North-East- and South-East-diagonals, i.e. $a_{i+1,j} \leq a_{i,j} \leq a_{i+1,j+1}$. An example of a Monotone Triangle of size 5 is given in Fig.1.

$$\begin{array}{ccccccccc} & & & & 4 & & & & \\ & & & 4 & & 5 & & & \\ & & 3 & & 5 & & 7 & & \\ & 2 & & 5 & & 6 & & 8 & \\ 2 & & 4 & & 5 & & 8 & & 9 \end{array}$$

Fig. 1: One of the 16939 Monotone Triangles with bottom row $(2, 4, 5, 8, 9)$.

For each $n \geq 1$, there exists a unique polynomial $\alpha(n; k_1, k_2, \ldots, k_n)$ of degree $n - 1$ in each of the n variables such that the evaluation of this polynomial at strictly increasing sequences $k_1 < k_2 < \cdots < k_n$ is equal to the number of Monotone Triangles with prescribed bottom row $(k_1, k_2, \ldots, k_n)$ – for example $\alpha(5; 2, 4, 5, 8, 9) = 16939$. This result was derived in [Fis06], where the polynomials are given explicitly in terms of an operator formula.

In [FR13] we studied the evaluation of $\alpha(n; k_1, \ldots, k_n)$ at weakly decreasing sequences $k_1 \geq k_2 \geq \cdots \geq k_n$. As it turned out, the evaluation can be interpreted as signed enumeration of the following combinatorial objects:

A *Decreasing Monotone Triangle* (DMT) of size n is a triangular array of integers $(a_{i,j})_{1 \leq j \leq i \leq n}$ having the following properties:

- The entries along NE- and SE-diagonals are weakly decreasing.
- Each integer appears at most twice in a row.
- Two consecutive rows do not both contain the same integer exactly once.

One of the motivations for considering evaluations of $\alpha(n; k_1, \ldots, k_n)$ at non-increasing $(k_1, \ldots, k_n) \in \mathbb{Z}^n$ stems from the connection to Alternating Sign Matrices. An *Alternating Sign Matrix* (ASM) of size n is a $n \times n$-matrix with entries in $\{0, 1, -1\}$ such that in each row and column the non-zero entries alternate in sign and sum up to 1. It is well-known ([MRR83]) that the set of ASMs is in bijection with the set of Monotone Triangles with bottom row $(1, 2, \ldots, n)$. Counting the number of ASMs of size n had been an open problem for more than a decade until the first two proofs were given by D. Zeilberger ([Zei96]) and G. Kuperberg ([Kup96]) in 1996 (see [Bre99] for more details). The Refined ASM Theorem – i.e. the refined enumeration with respect to the unique 1 in the first row – was reproven by I. Fischer in 2007 ([Fis07]). The identity

$$\alpha(n; k_1, \ldots, k_n) = (-1)^{n-1} \alpha(n; k_2, \ldots, k_n, k_1 - n) \tag{1}$$

plays one of the key roles in this algebraic proof. A bijective proof of (1) could give more combinatorial insight to the theorem. However, note that if $k_1 < k_2 < \cdots < k_n$, then $k_n > k_1 - n$, i.e. (1) can per se only be understood as identity satisfied by the polynomial.

The objective of this paper is to give a combinatorial interpretation to the evaluation of $\alpha(n; k_1, \ldots, k_n)$ at arbitrary $(k_1, \ldots, k_n) \in \mathbb{Z}^n$. For this, we define triangular arrays of integers which locally combine the restrictions of ordinary Monotone Triangles and Decreasing Monotone Triangles:

A *Generalized Monotone Triangle* (GMT) is a triangular array $(a_{i,j})_{1 \leq j \leq i \leq n}$ of integers satisfying the following conditions:

(1) Each entry is weakly bounded by its SW- and SE-neighbour, i.e.

$$\min\{a_{i+1,j}, a_{i+1,j+1}\} \leq a_{i,j} \leq \max\{a_{i+1,j}, a_{i+1,j+1}\}.$$

(2) If three consecutive entries in a row are weakly increasing, then their two interlaced neighbours in the row above are strictly increasing, i.e.

$$a_{i+1,j} \leq a_{i+1,j+1} \leq a_{i+1,j+2} \rightarrow a_{i,j} < a_{i,j+1}.$$

(3) If two consecutive entries in a row are strictly decreasing and their interlaced neighbour in the row above is equal to its SW-/SE-neighbour, then the interlaced neighbour has a left/right neighbour and is equal to it, i.e.

$$a_{i,j} = a_{i+1,j} > a_{i+1,j+1} \rightarrow a_{i,j-1} = a_{i,j},$$

$$a_{i+1,j} > a_{i+1,j+1} = a_{i,j} \rightarrow a_{i,j+1} = a_{i,j}.$$

Note that by Condition (1) and (2) three consecutive entries in a row of a GMT can not coincide. By way of illustration, let us find all GMTs with bottom row $(4, 2, 1, 3)$: first, construct all possible penultimate rows (l_1, l_2, l_3). Condition (1) implies that $l_1 \in \{2, 3, 4\}$, Condition (3) further restricts it to $l_1 \in \{2, 3\}$. If on the one hand $l_1 = 2$, then Condition (3) forces $l_2 = 2$. The right-most entry l_3 is bounded by 1 and 3, but since $l_1 = l_2 = l_3 = 2$ can not occur, we have $l_3 \in \{1, 3\}$. If on the other hand $l_1 = 3$, then Condition (3) implies that $l_2 = l_3 = 1$. Continuing in the same way with all penultimate rows yields the four GMTs depicted in Fig.2.

$$\boxed{\begin{array}{ccccccc} & & & 2 & & & \\ & & 2 & & 2 & & \\ & 2 & & 2 & & 1 & \\ 4 & & 2 & & 1 & & 3 \end{array}} \quad \boxed{\begin{array}{ccccccc} & & & 2 & & & \\ & & 2 & & 3 & & \\ & 2 & & 2 & & 3 & \\ 4 & & 2 & & 1 & & 3 \end{array}} \quad \boxed{\begin{array}{ccccccc} & & & 3 & & & \\ & & 2 & & 3 & & \\ & 2 & & 2 & & 3 & \\ 4 & & 2 & & 1 & & 3 \end{array}} \quad \boxed{\begin{array}{ccccccc} & & & 1 & & & \\ & & 1 & & 1 & & \\ & 3 & & 1 & & 1 & \\ 4 & & 2 & & 1 & & 3 \end{array}}$$

Fig. 2: The four GMTs with bottom row $(4, 2, 1, 3)$.

For $k_1 < k_2 < \cdots < k_n$, the set of GMTs with bottom row $(k_1, \ldots, k_n)$ is equal to the set of Monotone Triangles with this bottom row: Every GMT with strictly increasing bottom row is by Conditions (1) and (2) a Monotone Triangle. Conversely, the weak increase along NE- and SE-diagonals of Monotone Triangles implies Condition (1) of GMTs, the strict increase along rows Condition (2), and the premise of (3) can not hold.

For $k_1 \geq k_2 \geq \cdots \geq k_n$, the set of GMTs with bottom row $(k_1, \ldots, k_n)$ is equal to the set of Decreasing Monotone Triangles with this bottom row: The NE- and SE-diagonals of every GMT with weakly decreasing bottom row are by Condition (1) weakly decreasing. This also implies a weak decrease along rows, and since three consecutive equal entries can not occur, each integer appears at most twice in a row. Furthermore, two consecutive rows can not both contain an integer exactly once due to Condition (3). Conversely, the weak decrease of DMTs along NE- and SE-diagonals implies Condition (1) and weak decrease along rows. Thus, the premise of (2) can only hold if three consecutive entries coincide, which is not admissible in DMTs. Finally, Condition (3) follows from the weak decrease along rows together with the condition that two consecutive rows do not both contain the same integer exactly once.

Therefore, Generalized Monotone Triangles are indeed a joint generalization of ordinary Monotone Triangles and Decreasing Monotone Triangles. The main result of the paper is that the evaluation of $\alpha(n; k_1, k_2, \dots, k_n)$ at integral values is a signed enumeration of the GMTs with bottom row $(k_1, k_2, \dots, k_n)$. The sign of a GMT is determined by the following two statistics:

1. An entry $a_{i,j}$ is called *newcomer* if $a_{i+1,j} > a_{i,j} > a_{i+1,j+1}$.
2. A pair (x, x) of two consecutive equal entries in a row is called *sign-changing*, if their interlaced neighbour in the row below is also equal to x.

In the following, let $\mathcal{G}_n(k_1, k_2, \dots, k_n)$ denote the set of GMTs with bottom row $(k_1, k_2, \dots, k_n)$.

Theorem 1 *Let $n \geq 1$ and $(k_1, k_2, \dots, k_n) \in \mathbb{Z}^n$. Then*

$$\alpha(n; k_1, k_2, \dots, k_n) = \sum_{A \in \mathcal{G}_n(k_1, \dots, k_n)} (-1)^{\mathrm{sc}(A)},$$

where $\mathrm{sc}(A)$ is the total number of newcomers and sign-changing pairs in A.

Applying Theorem 1 to our example in Fig.2 gives $\alpha(4; 4, 2, 1, 3) = -2$, since only the left-most GMT has an even number of sign-changes.

Theorem 1 is known to be true for strictly increasing sequences $k_1 < k_2 < \cdots < k_n$, as in this case the set $\mathcal{G}_n(k_1, \dots, k_n)$ is equal to the set of Monotone Triangles with bottom row $(k_1, k_2, \dots, k_n)$ and $\mathrm{sc}(A) = 0$ for every Monotone Triangle.

Lemma 3 of [FR13] implies the correctness of Theorem 1 for weakly decreasing bottom rows: In this case $\mathcal{G}_n(k_1, \dots, k_n)$ is equal to the set of DMTs with bottom row $(k_1, \dots, k_n)$ and the sc-functions coincide. K. Jochemko and R. Sanyal recently gave a proof of the theorem in this case from a geometric point of view ([JS12]).

In Section 2 we sketch a straight-forward inductive proof of Theorem 1 using a recursion satisfied by $\alpha(n; k_1, \dots, k_n)$ and case distinctions (more details in [Rie12]). In Section 3 a connection with a known generalization ([Fis11]) is established, which enables us to give a shorter, more subtle proof of Theorem 1. Apart from being a joint generalization of Monotone Triangles and DMTs, the newly introduced generalization is more reduced in the sense that fewer cancellations occur in the signed enumeration than in previously known generalizations. In Section 4 we apply the theorem to give a combinatorial proof of an identity satisfied by $\alpha(n; k_1, \dots, k_n)$ and provide a collection of open problems.

2 Summation Operator & Proof of Theorem 1

The number of Monotone Triangles with bottom row $(k_1, \dots, k_n)$ can be counted recursively by determining all admissible penultimate rows $(l_1, \dots, l_{n-1})$ and summing over the number of Monotone Triangles with these bottom rows. The polynomial $\alpha(n; k_1, \dots, k_n)$ hence satisfies

$$\alpha(n; k_1, \dots, k_n) = \sum_{\substack{(l_1, \dots, l_{n-1}) \in \mathbb{Z}^{n-1}, \\ k_1 \leq l_1 \leq k_2 \leq l_2 \leq \cdots \leq k_{n-1} \leq l_{n-1} \leq k_n, \\ l_i < l_{i+1}}} \alpha(n-1; l_1, \dots, l_{n-1}) \tag{2}$$

for all $k_1 < k_2 < \cdots < k_n$, $k_i \in \mathbb{Z}$. In fact ([Fis06]), one can define a summation operator $\sum\limits_{(l_1,\ldots,l_{n-1})}^{(k_1,\ldots,k_n)}$ for arbitrary $(k_1,\ldots,k_n) \in \mathbb{Z}^n$ such that

$$\alpha(n;k_1,\ldots,k_n) = \sum_{(l_1,\ldots,l_{n-1})}^{(k_1,\ldots,k_n)} \alpha(n-1;l_1,\ldots,l_{n-1}) \tag{3}$$

holds. This summation operator is defined recursively by

$$\begin{aligned}\sum_{(l_1,\ldots,l_{n-1})}^{(k_1,\ldots,k_n)} A(l_1,\ldots,l_{n-1}) := &\sum_{(l_1,\ldots,l_{n-2})}^{(k_1,\ldots,k_{n-1})} \sum_{l_{n-1}=k_{n-1}+1}^{k_n} A(l_1,\ldots,l_{n-2},l_{n-1}) \\ &+ \sum_{(l_1,\ldots,l_{n-2})}^{(k_1,\ldots,k_{n-2},k_{n-1}-1)} A(l_1,\ldots,l_{n-2},k_{n-1}), \quad n \geq 2,\end{aligned} \tag{4}$$

with $\sum\limits_{()}^{(k_1)} := \mathrm{id}$. Using induction, it is clear that the summation operators in (2) and (3) coincide for increasing sequences $k_1 < k_2 < \cdots < k_n$. In order to give a meaning to (4) for arbitrary $(k_1,\ldots,k_n) \in \mathbb{Z}^n$, we have to extend the definition of simple sums. Motivated by the formal identity $\sum_{i=a}^{b} f(i) = \sum_{i=a}^{\infty} f(i) - \sum_{i=b+1}^{\infty} f(i)$ for $a \leq b$, we define

$$\sum_{i=a}^{b} f(i) := \begin{cases} 0, & b = a-1, \\ -\sum\limits_{i=b+1}^{a-1} f(i), & b+1 \leq a-1. \end{cases} \tag{5}$$

To prove (3) for arbitrary $(k_1,\ldots,k_n) \in \mathbb{Z}^n$, let us first note that applying $\sum\limits_{(l_1,\ldots,l_{n-1})}^{(k_1,\ldots,k_n)}$ to a polynomial in $(l_1,\ldots,l_{n-1})$ yields a polynomial in $(k_1,\ldots,k_n)$: In the base case $n=2$, write the polynomial $p(l_1)$ in terms of the binomial basis $p(l_1) = \sum_{i=0}^{m-1} c_i \binom{l_1}{i}$. The polynomial $q(x) := \sum_{i=0}^{m-1} c_i \binom{x}{i+1}$ then satisfies $q(x+1) - q(x) = p(x)$. For integers $k_1 \leq k_2$, it follows that $\sum_{l_1=k_1}^{k_2} p(l_1) = q(k_2+1) - q(k_1)$, but this is by definition (5) true for arbitrary $k_1, k_2 \in \mathbb{Z}$. The inductive step is immediate using (4). Thus, we know that the right-hand side of (3) is a polynomial in $(k_1,\ldots,k_n)$ coinciding with the polynomial on the left-hand side whenever $k_1 < k_2 < \cdots < k_n$. Since a polynomial in n variables is uniquely determined by these values, it follows that (3) indeed holds. The same is true for the alternative recursive description

$$\begin{aligned}\sum_{(l_1,\ldots,l_{n-1})}^{(k_1,\ldots,k_n)} A(l_1,\ldots,l_{n-1}) = &\sum_{(l_1,\ldots,l_{n-2})}^{(k_1,\ldots,k_{n-1})} \sum_{l_{n-1}=k_{n-1}}^{k_n} A(l_1,\ldots,l_{n-2},l_{n-1}) \\ &- \sum_{(l_1,\ldots,l_{n-3})}^{(k_1,\ldots,k_{n-2})} A(l_1,\ldots,l_{n-3},k_{n-1},k_{n-1}), \quad n \geq 3.\end{aligned} \tag{6}$$

Lemma 1 *For $(k_1,\dots,k_n)\in\mathbb{Z}^n$ let $\mathcal{P}(k_1,\dots,k_n)$ denote the set of $(n-1)$-st rows of elements in $\mathcal{G}_n(k_1,k_2,\dots,k_n)$. Then every function $A(l_1,\dots,l_{n-1})$ satisfies*

$$\sum_{(l_1,\dots,l_{n-1})}^{(k_1,\dots,k_n)} A(l_1,\dots,l_{n-1}) = \sum_{(l_1,\dots,l_{n-1})\in\mathcal{P}(k_1,\dots,k_n)} (-1)^{\mathrm{sc}(\boldsymbol{k};\boldsymbol{l})} A(l_1,\dots,l_{n-1}), \quad n\ge 2,$$

where $\mathrm{sc}(\boldsymbol{k};\boldsymbol{l}) := \mathrm{sc}(k_1,\dots,k_n;l_1,\dots,l_{n-1})$ is the total number of newcomers and sign-changing pairs in $(l_1,\dots,l_{n-1})$.

It is instructive to see how the base case $n=2$ follows from (4), (5) and the definition of GMTs. In general, the set of admissible values for an entry l_i depends on its neighbours l_{i-1} and l_{i+1} as well as the four adjacent entries k_{i-1}, k_i, k_{i+1} and k_{i+2} in the row below – ordered

$$\begin{array}{ccccccc} & l_{i-1} & & l_i & & l_{i+1} & \\ k_{i-1} & & k_i & & k_{i+1} & & k_{i+2} \end{array}$$

– in the following way: If $k_{i-1} > l_{i-1} = k_i$, then the only admissible value is $l_i = k_i$. Symmetrically, if $k_{i+1} = l_{i+1} > k_{i+2}$, then $l_i = k_{i+1}$. Otherwise, l_i can take any value strictly between k_i and k_{i+1}. To determine whether $l_i = k_i$ is allowed, distinguish between $k_i > k_{i+1}$, $k_{i-1} > k_i \le k_{i+1}$ and $k_{i-1} \le k_i \le k_{i+1}$. If $k_i > k_{i+1}$, then $l_i = k_i$ is admissible, if and only if $l_{i-1} = k_i$. If $k_{i-1} > k_i \le k_{i+1}$, then $l_i = k_i$ is admissible. If $k_{i-1} \le k_i \le k_{i+1}$, then $l_i = k_i$ is admissible, if and only if $l_{i-1} < k_i$. Determining whether $l_i = k_{i+1}$ is admissible works symmetrically.

In order to prove Lemma 1 inductively, we hence have to distinguish between the cases $k_{n-1} \le k_n$ (Case 1) and $k_{n-1} > k_n$ (Case 2). Since a different behaviour occurs depending on whether l_{n-1} – the rightmost entry of the penultimate row – is equal to k_{n-1} or not, we have to consider sub-cases 1.1, 1.2 and 2.1, 2.2 respectively. Using Recursion (4) in Case 1 and Recursion (6) in Case 2, one can now give a straight-forward proof of the Lemma. The proof in full length can be found in [Rie12]. Theorem 1 is then an immediate consequence of (3) and Lemma 1:

$$\begin{aligned}
\alpha(n;k_1,\dots,k_n) &= \sum_{(l_1,\dots,l_{n-1})}^{(k_1,\dots,k_n)} \alpha(n-1;l_1,\dots,l_{n-1}) \\
&= \sum_{(l_1,\dots,l_{n-1})\in\mathcal{P}(k_1,\dots,k_n)} (-1)^{\mathrm{sc}(\boldsymbol{k};\boldsymbol{l})} \alpha(n-1;l_1,\dots,l_{n-1}) \\
&= \sum_{(l_1,\dots,l_{n-1})\in\mathcal{P}(k_1,\dots,k_n)} (-1)^{\mathrm{sc}(\boldsymbol{k};\boldsymbol{l})} \sum_{A\in\mathcal{G}_{n-1}(l_1,\dots,l_{n-1})} (-1)^{\mathrm{sc}(A)} = \sum_{A\in\mathcal{G}_n(k_1,\dots,k_n)} (-1)^{\mathrm{sc}(A)}.
\end{aligned}$$

3 Connection with different extension & Alternative proof

In [Fis11] four different combinatorial extensions of $\alpha(n;k_1,\dots,k_n)$ to all $(k_1,\dots,k_n)\in\mathbb{Z}^n$ are described. The idea behind all of them is to write the sum in (2) in terms of simple summations, i.e. summations as defined in (5). In the third extension this is based on the inclusion-exclusion principle: For

$k_1 < k_2 < \cdots < k_n$ let

$$\begin{aligned} M &:= \{(l_1,\dots,l_{n-1}) \in \mathbb{Z}^{n-1} \mid \forall j : k_j \le l_j \le k_{j+1} \ \wedge \ l_j < l_{j+1}\}, \\ A &:= \{(l_1,\dots,l_{n-1}) \in \mathbb{Z}^{n-1} \mid \forall j : k_j \le l_j \le k_{j+1}\}, \\ A_i &:= \{(l_1,\dots,l_{n-1}) \in \mathbb{Z}^{n-1} \mid \forall j : k_j \le l_j \le k_{j+1} \ \wedge \ l_{i-1} = k_i = l_i\}, \quad i = 2,\dots,n-1. \end{aligned}$$

From $k_i < k_{i+1}$ it follows that $A_i \cap A_{i+1} = \emptyset$, and thus we have for any function $f(\boldsymbol{l}) := f(l_1,\dots,l_{n-1})$ that

$$\sum_{\boldsymbol{l}\in M} f(\boldsymbol{l}) = \sum_{\boldsymbol{l}\in A} f(\boldsymbol{l}) - \sum_{i=2}^{n-1} \sum_{\boldsymbol{l}\in A_i} f(\boldsymbol{l}) + \sum_{\substack{2\le i_1<i_2\le n-1 \\ i_2\ne i_1+1}} \sum_{\boldsymbol{l}\in A_{i_1}\cap A_{i_2}} f(\boldsymbol{l}) - \sum_{\substack{2\le i_1<i_2<i_3\le n-1 \\ i_{j+1}\ne i_j+1}} \sum_{\boldsymbol{l}\in A_{i_1}\cap A_{i_2}\cap A_{i_3}} f(\boldsymbol{l}) \cdots, \quad (7)$$

which can be written in terms of simple sums as

$$\sum_{p\ge 0}(-1)^p \sum_{\substack{2\le i_1<i_2<\cdots<i_p\le n-1 \\ i_{j+1}\ne i_j+1}} \sum_{l_1=k_1}^{k_2} \sum_{l_2=k_2}^{k_3} \cdots \sum_{l_{i_1}-1=k_{i_1}}^{k_{i_1}} \sum_{l_{i_1}=k_{i_1}}^{k_{i_1}} \cdots \sum_{l_{i_p}-1=k_{i_p}}^{k_{i_p}} \sum_{l_{i_p}=k_{i_p}}^{k_{i_p}} \cdots \sum_{l_{n-1}=k_{n-1}}^{k_n} f(\boldsymbol{l}). \quad (8)$$

Applying (5), we can interpret (8) for arbitrary $(k_1,\dots,k_n) \in \mathbb{Z}^n$. Hence, let us show that

$$\alpha(n;k_1,\dots,k_n) = \sum_{p\ge 0}(-1)^p \sum_{\substack{2\le i_1<i_2<\cdots<i_p\le n-1 \\ i_{j+1}\ne i_j+1}} \sum_{l_1=k_1}^{k_2} \sum_{l_2=k_2}^{k_3} \cdots \sum_{l_{i_1}-1=k_{i_1}}^{k_{i_1}} \sum_{l_{i_1}=k_{i_1}}^{k_{i_1}} \cdots \sum_{l_{i_p}-1=k_{i_p}}^{k_{i_p}} \sum_{l_{i_p}=k_{i_p}}^{k_{i_p}} \cdots \sum_{l_{n-1}=k_{n-1}}^{k_n} \alpha(n-1;l_1,\dots,l_{n-1}) \quad (9)$$

holds for $(k_1,\dots,k_n) \in \mathbb{Z}^n$: The correctness for $k_1 < k_2 < \cdots < k_n$ is ensured by (2), (7) and (8). Similar to the proof of (3) for arbitrary $(k_1,\dots,k_n) \in \mathbb{Z}^n$, it suffices to note that the right-hand side of (9) is a polynomial in $k_1,\dots,k_n$ and thus uniquely determined by its evaluations at $k_1 < \ldots < k_n$.

As pointed out in [Fis11], we can give (9) a combinatorial meaning by interpreting $\alpha(n;k_1,\dots,k_n)$ as signed enumeration of the following combinatorial objects: In a triangular array $(a_{i,j})_{1\le j\le i\le n}$ of integers, let us call the entries $a_{i-1,j-1}$ and $a_{i-1,j}$ the parents of $a_{i,j}$. Among the entries $(a_{i,j})_{1<j<i\le n}$, there may be *special entries*. Special entries in the same row must not be adjacent (choosing these special entries corresponds to fixing the i_l's in (9)). The requirements for the entries are

- If $a_{i,j}$ is special, then $a_{i-1,j-1} = a_{i,j} = a_{i-1,j}$.
- If $a_{i,j}$ is not the parent of a special entry and $a_{i+1,j} \le a_{i+1,j+1}$, then $a_{i+1,j} \le a_{i,j} \le a_{i+1,j+1}$.
- If $a_{i,j}$ is not the parent of a special entry and $a_{i+1,j} > a_{i+1,j+1}$, then $a_{i+1,j+1} > a_{i,j} > a_{i+1,j}$. In this case $a_{i,j}$ is called *inversion*.

Let us denote by $\mathcal{T}_n(k_1,\ldots,k_n)$ the set of these objects with bottom row $(a_{n,1},\ldots,a_{n,n})=(k_1,\ldots,k_n)$. For $A\in\mathcal{T}_n(k_1,\ldots,k_n)$ let $s(A)$ be the total number of special entries and inversions. Using induction and (9) yields

$$\alpha(n;k_1,\ldots,k_n)=\sum_{A\in\mathcal{T}_n(k_1,\ldots,k_n)}(-1)^{s(A)}. \tag{10}$$

In the following, we give an alternative proof of Theorem 1 by finding cancellations occurring in (10). An advantage of removing these cancellations is that the notion of special entries will no longer be required. In fact, what we obtain after this reduction are exactly the GMTs. To be more concrete, we can eliminate those arrays $(a_{i,j})_{1\le j\le i\le n}$ violating the condition

$$a_{i,j-1}\le a_{i,j}\le a_{i,j+1}\ \rightarrow\ a_{i-1,j-1}<a_{i-1,j}$$

by using the following sign-reversing involution: find the minimal index i, and under those the minimal index j such that $a_{i,j-1}\le a_{i,j}\le a_{i,j+1}$ and $a_{i-1,j-1}=a_{i,j}=a_{i-1,j}$. If $a_{i,j}$ is special, then turn it non-special, and vice-versa. Note that the minimality ensures that if $a_{i,j}$ is not special, then the neighbours of $a_{i,j}$ are not special, i.e. turning $a_{i,j}$ special is admissible. It follows that

$$\alpha(n;k_1,\ldots,k_n)=\sum_{\substack{A\in\mathcal{T}_n(k_1,\ldots,k_n)\\ a_{i,j-1}\le a_{i,j}\le a_{i,j+1}\rightarrow a_{i-1,j-1}<a_{i-1,j}}}(-1)^{s(A)}.$$

Note that in this reduced set an entry $a_{i,j}$ is special if and only if $a_{i-1,j-1}=a_{i,j}=a_{i-1,j}$. Since special entries now correspond to sign-changing pairs and inversions to newcomers, the only remaining part for proving Theorem 1 is to show that

$$\mathcal{G}_n(k_1,\ldots,k_n)=\{A\in\mathcal{T}_n(k_1,\ldots,k_n):a_{i,j-1}\le a_{i,j}\le a_{i,j+1}\ \rightarrow\ a_{i-1,j-1}<a_{i-1,j}\},$$

where an entry $a_{i,j}$ is special if and only if $a_{i-1,j-1}=a_{i,j}=a_{i-1,j}$.

Let $A\in\mathcal{G}_n(k_1,\ldots,k_n)$. Then two adjacent special entries in a row would imply three consecutive equal entries in a row, in contradiction to Condition (2) of GMTs. If $a_{i,j}$ is special, then $a_{i-1,j-1}=a_{i,j}=a_{i-1,j}$ by definition. If $a_{i+1,j}\le a_{i+1,j+1}$, then $a_{i+1,j}\le a_{i,j}\le a_{i+1,j+1}$ by Condition (1) of GMTs. If $a_{i+1,j}>a_{i+1,j+1}$, then $a_{i+1,j}\ge a_{i,j}\ge a_{i+1,j+1}$ by Condition (1) of GMTs, and if $a_{i+1,j}$ and $a_{i+1,j+1}$ are neither special, Condition (3) of GMTs implies that $a_{i+1,j}>a_{i,j}>a_{i+1,j+1}$. We thus have $A\in\mathcal{T}_n(k_1,\ldots,k_n)$, and the additional property is exactly Condition (2) of GMTs.

Conversely, let $A\in\mathcal{T}_n(k_1,\ldots,k_n)$ such that $a_{i,j-1}\le a_{i,j}\le a_{i,j+1}$ implies $a_{i-1,j-1}<a_{i-1,j}$. Conditions (1) and (2) of GMTs are then trivially satisfied. If $a_{i,j}=a_{i+1,j}>a_{i+1,j+1}$, then $a_{i+1,j}$ is a special entry, and thus $a_{i,j}=a_{i+1,j}=a_{i,j-1}$. Symmetrically, if $a_{i+1,j}>a_{i+1,j+1}=a_{i,j}$, then $a_{i+1,j+1}$ is special, and thus $a_{i,j}=a_{i+1,j+1}=a_{i,j+1}$. In total, we have $A\in\mathcal{G}_n(k_1,\ldots,k_n)$.

4 Applications & Open Problems

With this generalization at hand, we can try to give a combinatorial interpretation to identities satisfied by $\alpha(n;k_1,\ldots,k_n)$. By way of illustration, take the identity

$$\begin{aligned}&\alpha(n;k_1,\ldots,k_{i-1},k_i,k_i+1,k_{i+2},\ldots,k_n)\\&=\alpha(n;k_1,\ldots,k_{i-1},k_i,k_i,k_{i+2},\ldots,k_n)+\alpha(n;k_1,\ldots,k_{i-1},k_i+1,k_i+1,k_{i+2},\ldots,k_n).\end{aligned} \tag{11}$$

A combinatorial proof of this identity in the case that $k_1 < k_2 < \cdots < k_i$ and $k_i + 1 < k_{i+2} < \cdots < k_n$ was given in [Fis11]. Using Theorem 1, we can now give a combinatorial proof for arbitrary $(k_1, \ldots, k_n) \in \mathbb{Z}^n$ by showing that there exists a sign-preserving bijection

$$\begin{aligned}&\mathcal{G}_n(k_1, \ldots, k_{i-1}, k_i, k_i + 1, k_{i+2}, \ldots, k_n)\\ &\qquad \leftrightarrow \mathcal{G}_n(k_1, \ldots, k_{i-1}, k_i, k_i, k_{i+2}, \ldots, k_n) \mathbin{\dot\cup} \mathcal{G}_n(k_1, \ldots, k_{i-1}, k_i + 1, k_i + 1, k_{i+2}, \ldots, k_n).\end{aligned}$$

If $\mathcal{P}(k_1, \ldots, k_n)$ denotes the set of penultimate rows of GMTs with bottom row $(k_1, \ldots, k_n)$, it suffices to show that

$$\begin{aligned}&\mathcal{P}(k_1, \ldots, k_{i-1}, k_i, k_i + 1, k_{i+2}, \ldots, k_n)\\ &\qquad = \mathcal{P}(k_1, \ldots, k_{i-1}, k_i, k_i, k_{i+2}, \ldots, k_n) \mathbin{\dot\cup} \mathcal{P}(k_1, \ldots, k_{i-1}, k_i + 1, k_i + 1, k_{i+2}, \ldots, k_n), \qquad (12)\end{aligned}$$

where each fixed row has the same total number of sign-changes on both sides.

Each $(l_1, \ldots, l_{n-1}) \in \mathcal{P}(k_1, \ldots, k_{i-1}, k_i, k_i + 1, k_{i+2}, \ldots, k_n)$ satisfies $l_i \in \{k_i, k_i + 1\}$. Let us show that the set of penultimate rows with $l_i = k_i$ is equal to $\mathcal{P}(k_1, \ldots, k_{i-1}, k_i, k_i, k_{i+2}, \ldots, k_n)$. With $l_i = k_i$ it is clear that the restrictions for $(l_1, \ldots, l_{i-1})$ and $(l_{i+3}, \ldots, l_{n-1})$ are identical for both $\mathcal{P}(k_1, \ldots, k_{i-1}, k_i, k_i + 1, k_{i+2}, \ldots, k_n)$ and $\mathcal{P}(k_1, \ldots, k_{i-1}, k_i, k_i, k_{i+2}, \ldots, k_n)$. For the restrictions of (l_{i+1}, l_{i+2}) distinguish between $k_i + 1 \leq k_{i+2}$, $k_i = k_{i+2}$ and $k_i > k_{i+2}$:

- If $k_i + 1 \leq k_{i+2}$, then $k_i + 1 \leq l_{i+1} \leq k_{i+2}$ on both sides and the restrictions for l_{i+2} are the same:

Left-hand side of (12)	Right-hand side of (12)
$\begin{array}{ccccccccc} & & k_i & & & & l_{i+1} & & \\ & \rotatebox{45}{$=$} & & \rotatebox{-45}{$<$} & & \rotatebox{45}{$\leq$} & & \rotatebox{-45}{$\leq$} & \\ k_i & & < & & k_i+1 & & \leq & & k_{i+2} \end{array}$	$\begin{array}{ccccccccc} & & k_i & & & & l_{i+1} & & \\ & \rotatebox{45}{$=$} & & \rotatebox{-45}{$=$} & & \rotatebox{45}{$\leq$} & & \rotatebox{-45}{$\leq$} & \\ k_i & & = & & k_i & & \leq & & k_{i+2} \end{array}$

- If $k_i = k_{i+2}$, then $\mathcal{P}(k_1, \ldots, k_{i-1}, k_i, k_i, k_{i+2}, \ldots, k_n)$ is empty, and each element of $\mathcal{P}(k_1, \ldots, k_{i-1}, k_i, k_i + 1, k_{i+2}, \ldots, k_n)$ with $l_i = k_i$ would have to satisfy $l_i = l_{i+1} = l_{i+2} = k_i$:

Left-hand side of (12)	Right-hand side of (12)
$\begin{array}{cccccccccc} & & k_i & & & & k_i & & & k_i \\ & \rotatebox{45}{$=$} & & \rotatebox{-45}{$<$} & & \rotatebox{45}{$>$} & & \rotatebox{-45}{$=$} & & \\ k_i & & < & & k_i+1 & & > & & k_i & \end{array}$	$\begin{array}{ccccccccc} & & k_i & & & & \lightning & & \\ & \rotatebox{45}{$=$} & & \rotatebox{-45}{$=$} & & & & & \\ k_i & & = & & k_i & & = & & k_i \end{array}$

  But, since a GMT can not contain three consecutive equal entries, there is also no element in $\mathcal{P}(k_1, \ldots, k_{i-1}, k_i, k_i + 1, k_i, \ldots, k_n)$ with $l_i = k_i$.

- If $k_i > k_{i+2}$, then $k_i \geq l_{i+1} \geq k_{i+2}$ on both sides and the restrictions for l_{i+2} are the same:

Left-hand side of (12)	Right-hand side of (12)
$\begin{array}{ccccccccc} & & k_i & & & & l_{i+1} & & \\ & \rotatebox{45}{$=$} & & \rotatebox{-45}{$<$} & & \rotatebox{45}{$>$} & & \rotatebox{-45}{$\geq$} & \\ k_i & & < & & k_i+1 & & > & & k_{i+2} \end{array}$	$\begin{array}{ccccccccc} & & k_i & & & & l_{i+1} & & \\ & \rotatebox{45}{$=$} & & \rotatebox{-45}{$=$} & & \rotatebox{45}{$\geq$} & & \rotatebox{-45}{$\geq$} & \\ k_i & & = & & k_i & & > & & k_{i+2} \end{array}$

The entry l_{i+1} is involved in a sign-change on both sides (note the special case $l_{i+1} = k_i$, where l_{i+1} is a newcomer on the left-hand side and in a sign-changing pair on the right-hand side).

Symmetrically, one can also see that the set $\mathcal{P}(k_1, \dots, k_{i-1}, k_i, k_i + 1, k_{i+2}, \dots, k_n)$ restricted to $l_i = k_i + 1$ is the same as $\mathcal{P}(k_1, \dots, k_{i-1}, k_i + 1, k_i + 1, k_{i+2}, \dots, k_n)$, concluding the combinatorial proof of (11) for arbitrary $(k_1, \dots, k_n) \in \mathbb{Z}^n$.

A natural question is now whether similar identities hold if the difference between k_{i+1} and k_i is larger. For fixed integers $k_1, \dots, k_{i-1}, k_{i+2}, \dots, k_n$, let

$$t_n(k_i, k_{i+1}) := \alpha(n; k_1, \dots, k_{i-1}, k_i, k_{i+1}, k_{i+2}, \dots, k_n).$$

Similarly - with a bit more patience - one can also show the identity

$$t_n(k_i, k_i+2) = t_n(k_i, k_i)+t_n(k_i+1, k_i+1)+t_n(k_i+2, k_i+2)+t_n(k_i+2, k_i+1)+t_n(k_i+1, k_i) \quad (13)$$

combinatorially. Both (11) and (13) are special cases of the following algebraic identity: Let $V_{x,y}$ be the operator defined as $V_{x,y} f(x, y) := f(x - 1, y) + f(x, y + 1) - f(x - 1, y + 1)$. The function $f_i(k_1, \dots, k_n) := V_{k_i,k_{i+1}} \alpha(n; k_1, \dots, k_n)$ then satisfies

$$f_i(k_1, \dots, k_n) = -f_i(k_1, \dots, k_{i-1}, k_{i+1} + 1, k_i - 1, k_{i+2}, \dots, k_n). \quad (14)$$

Setting $k_{i+1} = k_i - 1$ in (14) immediately implies (11). Equation (13) is then the special case $k_{i+1} = k_i - 2$ in (14). A similar shift-antisymmetry property for *Gelfand-Tsetlin Patterns* (Monotone Triangles without the condition of strict increase along rows) was shown bijectively in a recent work ([Fis11]). It would be interesting to give a bijective proof of (14) in the general case (an algebraic proof was given in [Fis06]).

In [FR13] we showed the surprising identity

$$A_n := \alpha(n; 1, 2, \dots, n) = \alpha(2n; n, n, n - 1, n - 1, \dots, 1, 1) \quad (15)$$

algebraically and gave initial thoughts on how a bijective proof could succeed. Let us conclude with a list of related identities – all of them are up to this point conjectured using mathematical computing software. As Theorem 1 provides a combinatorial interpretation of these identities, bijective proofs are of high interest.

Conjecture 1 ([FR13]) *Let* $n \geq 1$. *Then*

$$\alpha(n; 2, 4, \dots, 2n) = (-1)^n \alpha(2n + 1; 2n + 1, 2n, \dots, 1) \quad (16)$$

holds, whereby the left-hand side is known to be the number of Vertically Symmetric ASMs of size $2n + 1$. *By Theorem 1, the right-hand side is further equal to* $\alpha(2n; 2n, 2n, 2n - 2, 2n - 2, \dots, 2, 2)$.

Conjecture 2 *Let* $n \geq 1$. *Then*

$$A_n = \alpha(n + i; 1, 2, \dots, i, 1, 2, \dots, n), \quad i = 0, \dots, n, \quad (17)$$

$$A_n = (-1)^n \alpha(2n + 1; 1, 2, \dots, n + 1, 1, 2, \dots, n) \quad (18)$$

holds. Furthermore, the numbers

$$W_{n,i} = \alpha(2n + 1; i, 2, \dots, n + 1, 1, 2, \dots, n), \quad i = 1, \dots, 3n + 2$$

satisfy the symmetry $W_{n,i} = W_{n,3n+3-i}$.

Conjecture 3 *Let $n \geq 2$. Then*

$$A_n = \alpha(n+2; 1, 2, \dots, i+1, i, i+1, \dots, n), \quad i = 1, \dots, n-1 \tag{19}$$

holds.

Further computational experiments led to the conjecture that (15) and (19) have the following joint generalization:

Conjecture 4 *Let $n \geq 1$. Then*

$$A_n = \alpha(n+k; 1, \dots, i-1, i+k-1, i+k-1, i+k-2, i+k-2, \dots, i, i, i+k, i+k+1, \dots, n) \tag{20}$$

holds for $i = 1, \dots, n-k+1$, $k = 1, \dots, n$.

In words, the last identity takes a subsequence $(i, i+1, \dots, i+k-1)$ of length k of $(1, 2, \dots, n)$, reverses the order, duplicates each entry and puts the subsequence back. Identity (15) is thus the special case of (20) where $k = n$. Applying (11) and the fact that a GMT can not contain three consecutive equal entries, shows that (19) is the special case of (20) with $k = 2$:

$$\begin{aligned}
&\alpha(n+2; 1, 2, \dots, i-1, i, i+1, i, i+1, i+2, \dots, n)\\
&= \alpha(n+2; 1, 2, \dots, i-1, i, i, i, i+1, i+2, \dots, n) + \alpha(n+2; 1, 2, \dots, i-1, i+1, i+1, i, i+1, i+2, \dots, n)\\
&= \alpha(n+2; 1, 2, \dots, i-1, i+1, i+1, i, i, i+2, \dots, n) + \alpha(n+2; 1, 2, \dots, i-1, i+1, i+1, i+1, i+1, i+2, \dots, n)\\
&= \alpha(n+2; 1, 2, \dots, i-1, i+1, i+1, i, i, i+2, \dots, n).
\end{aligned}$$

From the correspondence between ASMs of size n and Monotone Triangles with bottom row $(1, 2, \dots, n)$, it follows that $\alpha(n-1; 1, 2, \dots, i-1, i+1, \dots, n)$ is equal to the number of ASMs of size n with the first row's unique 1 in column i – denoted $A_{n,i}$. In the following conjecture we analogously remove the i-th argument of the right-hand side in (19):

Conjecture 5 *Let $n \geq 1$. Then*

$$\alpha(n+1; 1, 2, \dots, i-1, i+1, i, i+1, \dots, n) = -\sum_{j=1}^{n} (j-i) A_{n,j}, \quad i = 1, \dots, n-1 \tag{21}$$

holds.

As a note on how we found (21), let us prove the case $i = 1$: Each penultimate row $(l_1, \dots, l_n)$ of a GMT with bottom row $(2, 1, 2, \dots, n)$ satisfies $l_1 = l_2 = 1$ by Condition (3) of GMTs. Taking Conditions (1) and (2) into account, Lemma 1 implies that

$$\alpha(n+1; 2, 1, 2, \dots, n) = -\sum_{p=2}^{n} \alpha(n; 1, 1, 2, \dots, p-1, p+1, \dots, n).$$

Each penultimate row $(m_1, \dots, m_{n-1})$ of a GMT with bottom row $(1, 1, 2, \dots, p-1, p+1, \dots, n)$ satisfies $m_1 = 1, m_2 = 2, \dots, m_{p-1} = p-1$. Applying Lemma 1 again yields the claimed equation:

$$\alpha(n+1; 2, 1, 2, \dots, n) = -\sum_{p=2}^{n} \sum_{j=p}^{n} A_{n,j} = -\sum_{j=2}^{n} (j-1) A_{n,j}.$$

For general i, the set of GMTs with bottom row $(1, 2, \dots, i-1, i+1, i, i+1, \dots, n)$ can be written as disjoint union of those with structure

$$S_1: \begin{array}{ccccccccccccccccccc} & l_1 & & \cdots & & l_{i-2} & & i+1 & & i+1 & & i & & l_{i+2} & & \cdots & & l_n & \\ 1 & & \cdots & & i-2 & & i-1 & & i+1 & & i & & i+1 & & i+2 & & \cdots & & n, \end{array}$$

$$S_2: \begin{array}{ccccccccccccccccccc} & l_1 & & \cdots & & l_{i-2} & & i+1 & & i & & i & & l_{i+2} & & \cdots & & l_n & \\ 1 & & \cdots & & i-2 & & i-1 & & i+1 & & i & & i+1 & & i+2 & & \cdots & & n, \end{array}$$

$$S_3: \begin{array}{ccccccccccccccccccc} & l_1 & & \cdots & & l_{i-2} & & i-1 & & i & & i & & l_{i+2} & & \cdots & & l_n & \\ 1 & & \cdots & & i-2 & & i-1 & & i+1 & & i & & i+1 & & i+2 & & \cdots & & n. \end{array}$$

Similar to the case $i = 1$, one can see that the signed enumeration of GMTs with structure S_3 is equal to $-\sum_{j=i+1}^{n}(j-i)A_{n,j}$. Proving that the signed enumeration of GMTs with structure S_1 and S_2 yields $-\sum_{j=1}^{i-1}(j-i)A_{n,j}$ remains an open problem. A list of more conjectures can be found in [Rie12].

References

[Bre99] D. M. Bressoud. *Proofs and Confirmations: The Story of the Alternating-Sign Matrix Conjecture*. Cambridge University Press, 1999.

[Fis06] I. Fischer. The number of monotone triangles with prescribed bottom row. *Adv. Appl. Math.*, 37:249–267, 2006.

[Fis07] I. Fischer. A new proof of the refined alternating sign matrix theorem. *J. Comb. Theory Ser. A*, 114:253–264, 2007.

[Fis11] I. Fischer. Sequences of labeled trees related to Gelfand-Tsetlin patterns. *arXiv:1104.0568v1*, 2011.

[FR13] I. Fischer and L. Riegler. Combinatorial reciprocity for monotone triangles. *J. Comb. Theory Ser. A (accepted)*, 2013.

[JS12] K. Jochemko and R. Sanyal. Arithmetic of marked poset polytopes, monotone triangle reciprocity, and partial colorings. *arXiv:1206.4066*, 2012.

[Kup96] G. Kuperberg. Another proof of the alternating sign matrix conjecture. *Int. Math. Res. Notices*, 3:139–150, 1996.

[MRR83] W.H. Mills, D.P. Robbins, and H. Rumsey. Alternating sign matrices and descending plane partitions. *J. Comb. Theory Ser. A*, 34:340–359, 1983.

[Rie12] L. Riegler. Generalized monotone triangles: an extended combinatorial reciprocity theorem. *arXiv:1207.4437*, 2012.

[Zei96] D. Zeilberger. Proof of the alternating sign matrix conjecture. *Electron. J. Comb.*, 3:1–84, 1996.