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A direct bijection between permutations and a subclass of totally symmetric self-complementary plane partitions

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Abstract. We define a subclass of totally symmetric self-complementary plane partitions (TSSCPPs) which we show is in direct bijection with permutation matrices. This bijection maps the inversion number of the permutation, the position of the 1 in the last column, and the position of the 1 in the last row to natural statistics on these TSSCPPs. We also discuss the possible extension of this approach to finding a bijection between alternating sign matrices and all TSSCPPs. Finally, we remark on a new poset structure on TSSCPPs arising from this perspective which is a distributive lattice when restricted to permutation TSSCPPs.

Résumé. Nous définissons une sous-classe de partitions planes totalement symétriques autocomplémentaires (TSSCPPs) que nous montrons est en bijection directe avec des matrices permutation. Cette bijection trace le numéro inverse de la permutation, la position du 1 dans la derniére colonne, et la position du 1 dans le dernier rayon aux statistiques naturelles sur cettes TSSCPPs. Aussi, nous discutons l'extension possible de cette approche pour trouver une bijection entre les matrices á signe alternat et toutes TSSCPPs. Finalement, nous remarquons sur une structure poset nouvelle sur les TSSCPPs se levant de cette perspective qui est une treillis distributif quand elle est limité aux TSSCPPs permutation.

Keywords: alternating sign matrix, plane partition, permutation, bijection

1 Introduction

Alternating sign matrices (ASMs) and their equinumerous friends, descending plane partitions (DPPs) and totally symmetric self-complementary plane partitions (TSSCPPs), have been bothering combinatorialists for decades by the lack of an explicit bijection between any two of the three sets of objects. (See [7] [8] [1] [12] [6] for these enumerations and bijective conjectures and [4] for the story behind these papers.) In [9], we gave a bijection between permutation matrices (which are a subclass of ASMs) and descending plane partitions with no special parts in such a way that the inversion number of the permutation matrix equals the number of parts of the DPP. In this paper, we complete the solution to this bijection problem in the special case of permutations by identifying the subclass of TSSCPPs corresponding to permutations and giving a bijection which yields a direct interpretation for the inversion number on these permutation TSSCPPs.

In Section 2, we define TSSCPPs and ASMs and give bijections within their respective families. We recall the standard bijection from ASMs to monotone triangles. We then outline a known bijection from TSSCPPs to non-intersecting lattice paths and then transform these to new objects we call *boolean triangles*.

In Section 3, we identify the permutation subclass of TSSCPPs in terms of the boolean triangles of Section 2. We use this characterization to present a direct bijection between this subclass of TSSCPPs and permutation matrices. This bijection gives a natural interpretation on the TSSCPP for the inversions of the permutation as well as the positions of the 1's in the bottom row and last column of the permutation matrix.

It is not obvious how to extend this bijection to all ASMs and TSSCPPs. No one knows statistics on TSSCPPs with distributions corresponding to the inversion number or the number of -1's in an ASM. In Section 4, we discuss the outlook of the general bijection problem and compare the bijection of this paper with another recent bijection of Biane and Cheballah [3].

Finally, in Section 5 we make some remarks about a new partial order on TSSCPPs obtained via boolean triangles, which reduces in the permutation case to the distributive lattice which is the product of chains of lengths $2, 3, \ldots, n$.

2 The objects and their alter egos: ASMs & monotone triangles, TSSCPPs & non-intersecting lattice paths / boolean triangles

We first define ASMs and recall the standard bijection to monotone triangles. We then define TSSCPPs and give bijections with non-intersecting lattice paths and new objects we call boolean triangles. Then in the next section, we give a bijection from permutation ASMs to permutation TSSCPPs via these intermediary objects.

Definition 1 *An* alternating sign matrix *(ASM) is a square matrix with entries 0, 1, or* -1 *whose rows and columns each sum to 1 and such that the nonzero entries in each row and column alternate in sign.*

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

Fig. 1: The seven 3×3 ASMs.

See Figure 1 for the seven 3×3 ASMs. It is clear that the alternating sign matrices with no -1 entries are the permutation matrices.

Alternating sign matrices are known to be in bijection with monotone triangles, which are certain semistandard Young tableaux (that are also strict Gelfand-Tsetlin patterns). See Figure 2.

Definition 2 *A* monotone triangle *of order n is a triangular arrays of integers with i integers in row i for all $1 \leq i \leq n$, bottom row $1\ 2\ 3\ \cdots\ n$, and integer entries $a_{i,j}$ for $1 \leq i \leq n$, $n-i \leq j \leq n-1$ such that $a_{i,j-1} \leq a_{i-1,j} \leq a_{i,j}$ and $a_{i,j} < a_{i,j+1}$.*

It is well-known that monotone triangles of order n are in bijection with $n \times n$ alternating sign matrices via the following map [4]. For each row of the ASM note which columns have a partial sum (from the top) of 1 in that row. Record the numbers of the columns in which this occurs in increasing order. This process

$$\begin{array}{ccccc} & & 1 & & \\ & 1 & & 2 & \\ 1 & & 2 & & 3 \end{array} \quad \begin{array}{ccccc} & & 1 & & \\ & 1 & & 3 & \\ 1 & & 2 & & 3 \end{array} \quad \begin{array}{ccccc} & & 2 & & \\ & 1 & & 2 & \\ 1 & & 2 & & 3 \end{array} \quad \begin{array}{ccccc} & & 2 & & \\ & 1 & & 3 & \\ 1 & & 2 & & 3 \end{array} \quad \begin{array}{ccccc} & & 2 & & \\ & 2 & & 3 & \\ 1 & & 2 & & 3 \end{array} \quad \begin{array}{ccccc} & & 3 & & \\ & 1 & & 3 & \\ 1 & & 2 & & 3 \end{array} \quad \begin{array}{ccccc} & & 3 & & \\ & 2 & & 3 & \\ 1 & & 2 & & 3 \end{array}$$

Fig. 2: The seven monotone triangles of order 3, listed in order corresponding to Figure 1.

yields a monotone triangle of order n. Note that entries $a_{i,j}$ in the monotone triangle satisfying the strict diagonal inequalities $a_{i,j-1} < a_{i-1,j} < a_{i,j}$ are in bijection with the -1 entries of the corresponding ASM. Also, recall that the *inversion number* of an ASM A is defined as $I(A) = \sum A_{ij}A_{k\ell}$ where the sum is over all i, j, k, ℓ such that $i > k$ and $j < \ell$. This definition extends the usual notion of inversion in a permutation matrix.

We now define plane partitions.

Definition 3 *A* plane partition *is a two dimensional array of positive integers which weakly decreases across rows from left to right and down columns.*

We can visualize a plane partition as a stack of unit cubes pushed up against the corner of a room. If we identify the corner of the room with the origin and the room with the positive orthant, then denote each unit cube by its coordinates in $\mathbb{N}^3$, we obtain the following equivalent definition. A plane partition π is a finite set of positive integer lattice points (i, j, k) such that if $(i, j, k) \in \pi$ and $1 \le i' \le i$, $1 \le j' \le j$, and $1 \le k' \le k$ then $(i', j', k') \in \pi$. A plane partition is *totally symmetric* if whenever $(i, j, k) \in \pi$ then all six permutations of (i, j, k) are also in π.

Definition 4 *A* totally symmetric self–complementary plane partition *(TSSCPP) inside a $2n \times 2n \times 2n$ box is a totally symmetric plane partition which is equal to its complement, that is, the collection of empty cubes in the box is of the same shape as the collection of cubes in the plane partition itself.*

$$\begin{array}{cccccc} 6&6&6&3&3&3\\ 6&6&6&3&3&3\\ 6&6&6&3&3&3\\ 3&3&3&&&\\ 3&3&3&&&\\ 3&3&3&&& \end{array} \qquad \begin{array}{cccccc} 6&6&6&4&3&3\\ 6&6&5&3&3&2\\ 6&6&6&3&3&3\\ 4&3&3&1&&\\ 3&3&3&&&\\ 3&3&2&&& \end{array} \qquad \begin{array}{cccccc} 6&6&6&4&3&3\\ 6&6&5&4&3&3\\ 6&6&4&3&2&2\\ 4&4&3&2&&\\ 3&3&2&&&\\ 3&3&2&&& \end{array} \qquad \begin{array}{cccccc} 6&6&6&5&5&3\\ 6&5&5&4&3&1\\ 6&5&4&3&2&1\\ 5&4&3&2&1&\\ 5&3&2&1&1&\\ 3&1&1&&& \end{array}$$

$$\begin{array}{cccccc} 6&6&6&5&5&3\\ 6&5&5&3&3&1\\ 6&5&5&3&3&1\\ 5&3&3&1&1&\\ 5&3&3&1&1&\\ 3&1&1&&& \end{array} \qquad \begin{array}{cccccc} 6&6&6&5&4&3\\ 6&6&5&3&3&2\\ 6&5&5&3&3&1\\ 5&3&3&1&1&\\ 4&3&3&1&&\\ 3&2&1&&& \end{array} \qquad \begin{array}{cccccc} 6&6&6&5&4&3\\ 6&6&5&4&3&2\\ 6&5&4&3&2&1\\ 5&4&3&2&1&\\ 4&3&2&1&&\\ 3&2&1&&& \end{array}$$

Fig. 3: TSSCPPs inside a $6 \times 6 \times 6$ box

See Figure 3 for the seven TSSCPPs of order 3.

In [5], Di Francesco gives a bijection from TSSCPPs of order n to a collection of nonintersecting lattice paths. The bijection proceeds by taking a fundamental domain of the TSSCPP, and instead of reading the number of boxes in each stack, one looks at the paths going alongside those boxes. This yields a collection of nonintersecting paths with two types of steps. With a slight further deformation, he obtains that the following objects are in bijection with TSSCPPs. See Figure 4.

Proposition 5 (Di Francesco) *Totally symmetric self-complementary plane partitions inside a $2n \times 2n \times 2n$ box are in bijection with nonintersecting lattice paths (NILP) starting at $(i, -i)$, $i = 1, 2, \ldots, n-1$, and ending at positive integer points on the x-axis of the form $(r_i, 0)$, $i = 1, 2, \ldots, n-1$, making only vertical steps $(0, 1)$ or diagonal steps $(1, 1)$.*

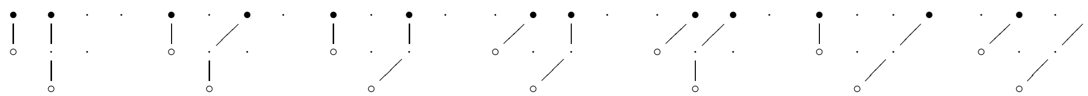


Fig. 4: The seven TSSCPP NILP of order 3.

In [5], Di Francesco uses the Lindström-Gessel-Viennot formula for counting nonintersecting lattice paths via a determinant evaluation to give an expression for the generating function of TSSCPPs with a weight of τ per vertical step. We will show that when restricted to permutation TSSCPPs, this weight corresponds to the inversion number of the permutation. Note that the distribution of the number of vertical steps in all TSSCPP NILPs does not correspond to the inversion number distribution on ASMs.

With another slight deformation, we obtain a tableaux version of these NILPs. See Figures 5 and 6.

Definition 6 *A* boolean triangle *of order n is a triangular integer array $\{b_{i,j}\}$ for $1 \leq i \leq n-1$, $n-i \leq j \leq n-1$ with entries in $\{0,1\}$ such that the diagonal partial sums satisfy*

$$1 + \sum_{i=j+1}^{i'} b_{i,n-j-1} \geq \sum_{i=j}^{i'} b_{i,n-j}. \tag{1}$$

$$\begin{array}{ccccccc}
 & & & b_{1,n-1} & & & \\
 & & b_{2,n-2} & & b_{2,n-1} & & \\
 & b_{3,n-3} & & b_{3,n-2} & & b_{3,n-1} & \\
 & & & \vdots & & & \\
b_{n-1,1} & & b_{n-1,2} & \cdots & b_{n-1,n-2} & & b_{n-1,n-1}
\end{array}$$

Fig. 5: A generic boolean triangle

Proposition 7 *Boolean triangles of order n are in bijection with TSSCPPs inside a $2n \times 2n \times 2n$ box.*

Proof: The bijection proceeds by replacing each vertical step of the NILP with a 1 and each diagonal step with a 0 and vertically reflecting the array. The inequality on the partial sums is equivalent to the condition that the lattice paths are nonintersecting. $\square$

1	1	0	0	1	0	0
1 1	1 0	1 1	0 1	0 0	1 0	0 0

Fig. 6: The seven TSSCPP boolean triangles of order 3, listed in order corresponding to Figure 4 (and Figure 2 via the bijection of Theorem 9).

3 A bijection on permutations

In this section, we give a bijection between $n \times n$ permutation matrices and a subclass of totally symmetric self-complementary plane partitions inside a $2n \times 2n \times 2n$ box, preserving the inversion number statistic and two boundary statistics. First, we identify the permutation subclass of TSSCPPs.

Definition 8 *Let* permutation TSSCPPs *of order n be all TSSCPPs of order n whose corresponding boolean triangles have weakly decreasing rows. (In the NILP picture, each row has some number of vertical steps followed by some number of diagonal steps.)*

It is easy to see that there are $n!$ permutation TSSCPPs. The condition on the boolean triangle that the rows be weakly decreasing means that all the 1's must be left-justified, thus the defining partial sum inequality (1) is never violated. To construct a permutation TSSCPP, freely choose any number of left-justified 1's in each row of the boolean triangle and the rest zeros; there are $i+1$ choices for row i, and the choices are all independent.

We are now ready to state and prove our main theorem.

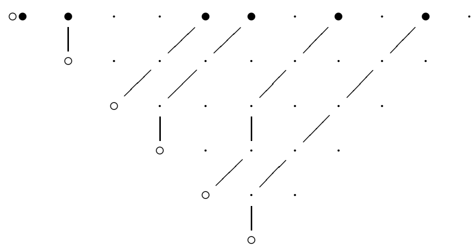
Theorem 9 *There is a natural, statistic-preserving bijection between $n \times n$ permutation matrices with inversion number p whose 1 in the last row is in column k and whose 1 in the last column is in row ℓ and permutation TSSCPPs of order n with p zeros in the boolean triangle, exactly $n-k$ of which are contained in the last row, and for which the lowest 1 in diagonal $n-1$ is in row $\ell-1$.*

Proof: We first describe the bijection map. An example of this bijection is shown in Figure 7.

Begin with a permutation TSSCPP of order n. Consider its associated boolean triangle $b = \{b_{i,j}\}$ for $1 \le i \le n-1$, $n-i \le j \le n-1$. Define $a = \{a_{i,j}\}$ for $1 \le i \le n$, $n-i \le j \le n-1$ as follows: $a_{n,j} = j+1$ and for $i < n$, $a_{i,j} = a_{i+1,j}$ if $b_{i,j} = 0$ and $a_{i,j} = a_{i+1,j-1}$ if $b_{i,j} = 1$. We claim a is a monotone triangle. Clearly $a_{i,j-1} \le a_{i-1,j} \le a_{i,j}$. Also, $a_{i,j} < a_{i,j+1}$, since if $a_{i,j} = a_{i,j+1}$, then $a_{i,j} = a_{i+1,j}$ and $a_{i,j+1} = a_{i+1,j+1}$ so that we would need $b_{i,j} = 0$ and $b_{i,j+1} = 1$. This contradicts the fact that the rows of permutation boolean triangles must weakly decrease. Furthermore, a is a monotone triangle with no -1's in the corresponding ASM, since each entry is defined to be equal to one of it's diagonal neighbors in the row below. This process is clearly invertible.

We now show that this map takes a permutation TSSCPP boolean triangle with p zeros to a permutation matrix with p inversions. Recall that the inversion number of any ASM A (with the matrix entry in row i and column j denoted A_{ij}) is defined as $I(A) = \sum A_{ij} A_{k\ell}$ where the sum is over all i, j, k, ℓ such that $i > k$ and $j < \ell$. This definition extends the usual notion of inversion in a permutation matrix. In [10] we found that $I(A)$ satisfies $I(A) = E(A) + N(A)$, where $N(A)$ is the number of -1's in A and $E(A)$ is the number of entries in the monotone triangle equal to their southeast diagonal neighbor (entries $a_{i,j}$ satisfying $a_{i,j} = a_{i+1,j}$). Since in our case, $N(A) = 0$ and $E(A)$ equals the number of zeros in the corresponding TSSCPP boolean triangle, we have that $I(A)$ equals the number of zeros in b.

TSSCPP



$\Leftrightarrow$

Boolean triangle

$$\begin{array}{ccccccccc} &&&&1&&&& \\ &&&0&&0&&& \\ &&1&&1&&0&& \\ &0&&0&&0&&0& \\ 1&&0&&0&&0&&0 \end{array}$$

Monotone triangle

$$\Leftrightarrow \quad \begin{array}{ccccccccccc} &&&&&4&&&&& \\ &&&&\mathbf{4}&&\mathbf{6}&&&& \\ &&&3&&4&&\mathbf{6}&&& \\ &&\mathbf{3}&&\mathbf{4}&&\mathbf{5}&&\mathbf{6}&& \\ &1&&\mathbf{3}&&\mathbf{4}&&\mathbf{5}&&\mathbf{6}& \\ 1&&2&&3&&4&&5&&6 \end{array}$$

Permutation matrix

$$\Leftrightarrow \begin{pmatrix} 0&0&0&1&0&0 \\ 0&0&0&0&0&1 \\ 0&0&1&0&0&0 \\ 0&0&0&0&1&0 \\ 1&0&0&0&0&0 \\ 0&1&0&0&0&0 \end{pmatrix}$$

Fig. 7: An example of the bijection. The bold entries in the monotone triangle are the entries equal to their southeast diagonal neighbor. These are exactly the diagonal steps of the TSSCPP. Note that the matrix on the right represents the permutation 463512 which has 11 inversions. These inversions correspond to the 11 diagonal steps of the TSSCPP on the left.

We can see that the zeros of b correspond to permutation inversions directly by noting that to convert from the monotone triangle representation of a permutation to a usual permutation σ such that $i \to \sigma(i)$, we set $\sigma(i)$ equal to the unique new entry in row i of the monotone triangle. Thus for each entry of the monotone triangle $a_{i,j}$ such that $a_{i,j} = a_{i+1,j}$, there will be an inversion in the permutation between $a_{i,j}$ and $\sigma(i+1)$. This is because $a_{i,j} = \sigma(k)$ for some $k \leq i$ and $\sigma(k) = a_{i,j} > \sigma(i)$. These entries $a_{i,j}$ such that $a_{i,j} = a_{i+1,j}$ correspond exactly to zeros in row i of the boolean triangle b. Thus if a permutation TSSCPP has p zeros in its boolean triangle, its corresponding permutation will have p inversions.

Also, observe that if the number of zeros in the last row of the boolean triangle is k, then the 1 in the bottom row of the permutation matrix will be in column $n-k$. So the missing number in the penultimate monotone triangle row shows where the last row of the boolean triangle transitions from ones to zeros. So by the bijection between monotone triangles and ASMs, the 1 in the last row of A is in column $n-k$.

Finally, if the lowest 1 in diagonal $n-1$ of the boolean triangle is in row $\ell-1$, this means that the entries $\{a_{i,n-1}\}$ for $\ell \leq i \leq n$ are all equal to n. So the 1 in the last column of the permutation matrix is in row ℓ. □

See Figure 7 for an example of this bijection.

4 Toward a bijection between all TSSCPPs and ASMs

In [9], we discussed the obstacles to turning the bijection between permutations and descending plane partitions presented there into a bijection between all ASMs and DPPs. Here we discuss some of the challenges to the ASM-TSSCPP bijection in full generality.

While DPPs have the property that the number of parts equals the inversion number of the ASM (this is now proved, though not bijectively [2]), TSSCPPs do not have such a statistic as of yet. We showed that the number of diagonal steps in a permutation-NILP gives the inversion number of the permutation matrix, but this is not true for general TSSCPPs and ASMs. Furthermore, while the number of special parts of a DPP corresponds to the number of -1's in the ASM, there is no such statistic on TSSCPP. It would seem reasonable to conjecture that the -1 of the ASM should correspond to all instances of a vertical step followed by a diagonal step as you go from left to right along a row of the NILP (or a 0 followed by a 1 as you go across a row of the boolean triangle). This holds up to $n = 4$, and it seems to hold for arbitrary n in the special cases of one -1 and the maximal number of -1's ($\lfloor \frac{n^2}{4} \rfloor$). But for the number of -1's between 1 and $\lfloor \frac{n^2}{4} \rfloor$, these statistics diverge.

Di Francesco has noted that the distribution of diagonal steps in the top row of the TSSCPP-NILP corresponds to the refined enumeration of ASMs. So one might hope to begin a general bijection by determining the $(n-1)$st row of the monotone triangle from the top row of the NILP (or the bottom row of the boolean triangle) by left-justifying all the vertical steps and then bijecting in the same way as in the permutation case. After that, though, it is unclear how to proceed. See Figure 4 for a summary of the various statistics which are preserved in the permutation case DPP-ASM-TSSCPP bijections and which should correspond in full generality. (See [9] for further explanation on the DPP case.)

DPP	ASM	TSSCPP boolean triangle
no special parts*	no -1's	rows weakly decrease
number of parts*	number of inversions	number of zeros
number of n's*	position of 1 in last column	position of lowest 1 in last diagonal
largest part value that does not appear	position of 1 in last row	number of zeros in last row*

Fig. 8: This table show the statistics preserved by the permutation case bijections of this paper and [9]. There is a star by the DPP and TSSCPP statistics that have the same distribution as the ASM statistic in the general case.

Finally, we compare this work with another recent bijection due to Biane and Cheballah. In [3], the authors give a bijection between *Gog* and *Magog trapezoids* of two diagonals. (*Gog triangles* are exactly monotone triangles. *Magog triangles* can be seen to be in bijection with the TSSCPP boolean triangles considered here. The term *trapezoid* indicates the truncation of the triangle to a fixed number of diagonals.) Their bijection is both more and less general than the one of this paper. It is more general in the sense that it includes configurations corresponding to the -1 in an ASM, where we consider only permutations. It is less general in that it uses only two diagonals of the triangle, where we are able to consider the full triangle.

Experimental evidence suggests the bijection of [3] and the bijection of this paper may coincide (up to slight deformation) in the case of permutation monotone triangles, truncated to two diagonals. Perhaps the combination of these two perspectives will provide insight on the full bijection.

5 Poset Structure

In [10], we examined a poset structure on TSSCPPs, which turned out to be a distributive lattice with poset of join irreducibles very similar to that of the ASM lattice. In this final section, we remark on a new partial order on TSSCPPs arising from this perspective which is not a distributive lattice, but which has nice distributive lattice structure when restricted to the permutation case.

Define the *boolean partial order* on TSSCPPs of order n as the boolean triangles of order n ordered by componentwise comparison of the entries. This is an induced subposet of the Boolean lattice on $\binom{n}{2}$ elements given by only taking the elements corresponding to TSSCPPs. This order on TSSCPPs is not a distributive lattice. But if we further restrict this order to the permutation TSSCPPs, the poset formed is $[2] \times [3] \times \cdots \times [n]$, that is, the product of chains of length $2, 3, 4, \ldots, n$, where the order ideal composed of k elements in the chain $[i]$ corresponds to row $i-1$ of the boolean triangle containing k 1's. This permutation TSSCPP lattice is a partial order on permutations which sits between the weak and strong Bruhat orders on the symmetric group. It contains all of the ordering relations of the weak order plus some of the additional relations of the strong order. See Figure 9.

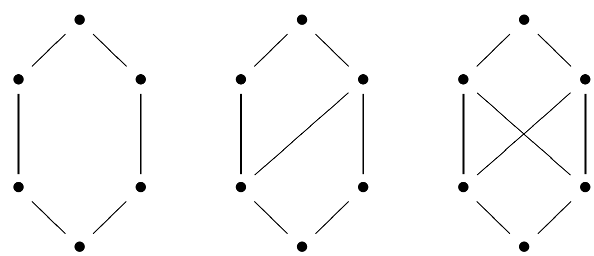


Fig. 9: From left to right: The weak order on S_3, the boolean partial order on permutation TSSCPPs of order 3, and the strong Bruhat order on S_3.

Conversely, the natural partial order on all ASMs is the distributive lattice of monotone triangles, but its restriction to permutations is the strong Bruhat order, which is not a lattice. In fact, the ASM lattice is the smallest lattice to contain the Bruhat order on the permutations as a subposet (i.e. it is the MacNeille completion of the Bruhat order [11]). See Figure 10 for a comparison of this order on ASMs with the TSSCPP boolean order.

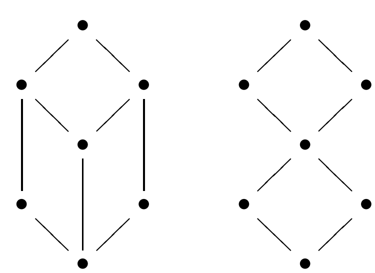


Fig. 10: Left: The boolean partial order on TSSCPPs of order 3. Right: The lattice of 3×3 ASMs.

We hope that the study of this new partial order on TSSCPPs will provide insight on the combinatorics of these objects and the associated outstanding bijection problems.

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