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Gelfand Models for Diagram Algebras: extended abstract

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Abstract. A Gelfand model for a semisimple algebra A over $\mathbb{C}$ is a complex linear representation that contains each irreducible representation of A with multiplicity exactly one. We give a method of constructing these models that works uniformly for a large class of combinatorial diagram algebras including: the partition, Brauer, rook monoid, rook-Brauer, Temperley-Lieb, Motzkin, and planar rook monoid algebras. In each case, the model representation is given by diagrams acting via "signed conjugation" on the linear span of their vertically symmetric diagrams. This representation is a generalization of the Saxl model for the symmetric group, and, in fact, our method is to use the Jones basic construction to lift the Saxl model from the symmetric group to each diagram algebra. In the case of the planar diagram algebras, our construction exactly produces the irreducible representations of the algebra.

Résumé. Un modèle de Gelfand pour une algèbre semi-simple A sur $\mathbb{C}$ est une représentation linéaire complexe qui contient chaque représentation irréductible de A avec multiplicité exactement un. Nous fournissons une méthode de construction explicite de ces modèles qui fonctionne de manière uniforme pour une grande classe d'algèbres de schéma combinatoire, y compris: la partition, Brauer, rook-monoid, rook-Brauer, Temperley-Lieb, Motzkin, et algèbres planaires rook monoid. En chaque cas, la représentation du modèle est donnée par les diagrammes agissant par "conjugaison signé" sur l'espace engendré par les diagrammes verticalement symétriques. Cette représentation est une généralisation du modèle Saxl pour le groupe symétrique, et, en fait, notre méthode est d'utiliser le "Jones basic construction" pour étendre le modèle Saxl du groupe symétrique à chaque algèbre diagramme. Dans le cas des algèbres de diagrammes planaires, notre construction produit exactement les représentations irréductibles de l'algèbre.

Keywords: Gelfand model; multiplicity-free representation; symmetric group; partition algebra; Brauer algebra; Temperley-Lieb algebra; Motzkin algebra; rook-monoid

1 Introduction

A famous consequence of Robinson-Schensted-Knuth (RSK) insertion is that the set of standard Young tableaux with k boxes is in bijection with the set of involutions in the symmetric group S_k (the permutations $\sigma \in \mathsf{S}_k$ with $\sigma^2 = 1$). Furthermore, these standard Young tableux index the bases for the irreducible $\mathbb{C}\mathsf{S}_k$ modules, so it follows that the sum of the degrees (dimensions) of the irreducible S_k modules equals the number of involutions in S_k. This suggests the possibility of a representation of the symmetric group

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on the linear span of its involutions which decomposes into irreducible S_k-modules such that the multiplicity of each irreducible is exactly 1. Indeed, Saxl [**?**] and Kljačko [**?**] have constructed such a module. In this representation, the symmetric group acts on its involutions by a twisted, or signed, conjugation (see Section 3). A combinatorial construction of this module was studied recently by Adin, Postnikov, and Roichman [**?**] and extended to the rook monoid and related semigroups in [**?**]. A representation for which each irreducible appears with multiplicity one is called a *Gelfand model* (or, simply, a *model*), because of the work in [**?**] on models for complex Lie groups.

In [**?**] the RSK algorithm is extended to work for a large class of well-known, combinatorial diagram algebras including the partition, Brauer, rook monoid, rook-Brauer, Temperley-Lieb, Motzkin, and planar rook monoid algebras. A consequence [**?**, (5.5)] of this algorithm is that the sum of the degrees of the irreducible representations of each of these algebras equals the number of horizontally symmetric basis diagrams in the algebra. This suggests the existence of a model representation of each of these algebras on the span of its symmetric diagrams, and the main result of this paper is to produce a such a model.

Let A_k denote one of the following unital associative $\mathbb{C}$-algebras: the partition, Brauer, rook monoid, rook-Brauer, Temperley-Lieb, Motzkin, or planar rook monoid algebra. Then A_k has a basis of diagrams and a multiplication given by diagram concatenation. The algebra A_k depends on a parameter $x \in \mathbb{C}$ and is semisimple for all but a finite number of choices of x. When A_k is semisimple, its irreducible modules are indexed by a set Λ_{A_k}, and for $\lambda \in \Lambda_{\mathsf{A}_k}$, we let A_k^λ denote the irreducible A_k-module labeled by λ. We construct, in a uniform way, an A_k-module $\mathsf{M}_{\mathsf{A}_k}$ which decomposes into irreducibles as $\mathsf{M}_{\mathsf{A}_k} \cong \bigoplus_{\lambda\in\Lambda_{\mathsf{A}_k}} \mathsf{A}_k^\lambda$, where the multiplicity of each irreducible module is exactly one.

Our model representation is constructed as follows. For a basis diagram d, we let d^T be its reflection across its horizontal axis and say that a diagram t is symmetric if $t^T = t$. A basis diagram d acts on a symmetric diagram t by "signed conjugation": $d \cdot t = \mathsf{sign}(d,t)\, dtd^T$, where $\mathsf{sign}(d,t)$ is the sign on the permutation of the fixed blocks of t induced by conjugation by d (see Section 4 for details). In each example, our basis diagrams are assigned a rank, which is the number of blocks in the diagram that propagate from the top row to the bottom row. We let $\mathsf{M}^r_{\mathsf{A}_k}$ be the linear span of the symmetric diagrams of rank r and our model is the direct sum $\mathsf{M}_{\mathsf{A}_k} = \oplus_{r=0}^k \mathsf{M}^r_{\mathsf{A}_k}$.

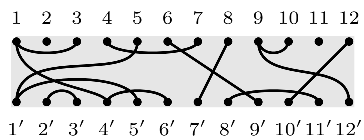
The diagram algebras in this paper naturally form a tower $\mathsf{A}_0 \subseteq \mathsf{A}_1 \subseteq \cdots \subseteq \mathsf{A}_k$, and we are able to use the structure of the Jones basic construction of this tower to derive our model. Each algebra contains a basic construction ideal $\mathsf{J}_{k-1} \subseteq \mathsf{A}_k$ such that $\mathsf{A}_k \cong \mathsf{J}_{k-1} \oplus \mathsf{C}_k$, where $\mathsf{C}_k \cong \mathbb{C}\mathsf{S}_k$ for nonplanar diagram algebras and $\mathsf{C}_k \cong \mathbb{C}\mathbf{1}_k$ for planar diagram algebras. The ideal J_{k-1} is in Schur-Weyl duality with one of A_{k-1} or A_{k-2} (depending on the specific diagram algebra). In this setup, we are able to take a model for each $\mathsf{C}_r, 0 \le r \le k$, and lift them to a module for A_k.

For the planar diagram algebras – the Temperley-Lieb, Motzkin, and planar rook monoid algebras – the algebra $\mathsf{C} \cong \mathbb{C}\mathbf{1}_k$ is trivial and the model is trivial. It follows that $\mathsf{M}^r_{\mathsf{A}_k}$ is irreducible and that signed conjugation produces a complete set of irreducible modules for the planar algebras. For the nonplanar diagram algebras, the algebra is $\mathsf{C} \cong \mathbb{C}\mathsf{S}_k$, and we use the Saxl model for S_r. In this case $\mathsf{M}^r_{\mathsf{A}_k}$ is further graded as $\mathsf{M}^r_{\mathsf{A}_k} = \oplus_f \mathsf{M}^{r,f}_{\mathsf{A}_k}$, where $\mathsf{M}^{r,f}_{\mathsf{A}_k}$ is the linear span of symmetric diagrams of rank r having f "fixed blocks" and $\mathsf{M}^{r,f}_{\mathsf{A}_k}$ decomposes into irreducibles labeled by partitions $\lambda \vdash r$ having f odd parts.

Besides being natural constructions, these model representations are useful in several ways. (1) In a model representation, isotypic components are irreducible components, so projection operators map directly onto irreducible modules without being mixed up among multiple isomorphic copies of the same module. (2) A key feature of our model is that we give the explicit action of each basis element of A_k on the basis of $\mathsf{M}^{r,f}_{\mathsf{A}_k}$. For small values of k, and for all values of k in the planar case, these representations are irreducible or have few irreducible components. Thus, in practice, the model provides a natural and easy way to compute the explicit action of basis diagrams on irreducible representations. (3) Gelfand models are useful in the study of Markov chains on related combinatorial objects; see, for example, Chapter 3F of [**?**] and the references therein, as well as [**?**], [**?**].

2 The Partition Algebra and its Diagram Subalgebras

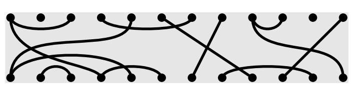
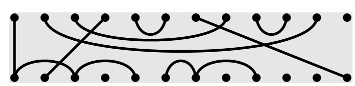
For $k \in \mathbb{Z}_{>0}$, let $\mathcal{P}_k$ denote the set of set partitions of $\{1, 2, \ldots, k, 1', 2', \ldots, k'\}$. We represent a set partition $d \in \mathcal{P}_k$ by a diagram with k vertices in the top row, labeled $1, \ldots, k$, and k vertices in the bottom row, labeled $1', \ldots, k'$. We then assign edges in this diagram so that its connected components equal the underlying set partition d. For example, the following is a diagram $d \in \mathcal{P}_{12}$,



$$= \left\{ \begin{array}{l} \{1,3,4',6'\}, \{2\}, \{4,7\}, \{5,1',5'\}, \{6,9'\}, \{8,7'\}, \\ \{9,10,12'\}, \{11\}, \{12,10'\}, \{2',3'\}, \{8',11'\} \end{array} \right\}.$$

We refer to the parts of a set partition as *blocks*, so that the above diagram has 11 blocks. The diagram of d is not unique, since it only depends on the underlying connected components.

Multiply two set partition diagrams $d_1, d_2 \in \mathcal{P}_k$ as follows. Place d_1 above d_2 and identify each vertex j' in the bottom row of d_1 with the corresponding vertex j in the top row of d_2. Remove any connected components that live entirely in the middle row and let $d_1 \circ d_2 \in \mathcal{P}_k$ be the resulting diagram. For example, if

$d_1 =$  and $d_2 =$ 

then

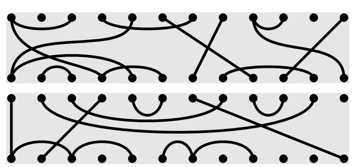
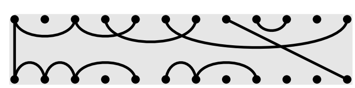
$d_1 \circ d_2 =$  $=$  .

Diagram multiplication is associative and makes $\mathsf{P}_k(x)$ a monoid with identity $\mathbf{1}_k =$  .

Now let $x \in \mathbb{C}$, define $\mathsf{P}_0(x) = \mathbb{C}$, and for $k \geq 1$, let $\mathsf{P}_k(x)$ be the $\mathbb{C}$-vector space with basis $\mathcal{P}_k$. If $d_1, d_2 \in \mathcal{P}_k$, let $\kappa(d_1, d_2)$ denote the number of connected components that are removed from the middle row in computing $d_1 \circ d_2$, and define

$$d_1 d_2 = x^{\kappa(d_1,d_2)}\, d_1 \circ d_2. \tag{1}$$

In the multiplication example of the previous section $\kappa(d_1, d_2) = 1$ and $d_1 d_2 = x(d_1 \circ d_2)$. This product makes $\mathsf{P}_k(x)$ an associative algebra with identity $\mathbf{1}_k$.

We say that a block B in a set partition diagram $d \in \mathcal{P}_k$ is a *propagating* block if B contains vertices from both the top and bottom row of d; that is, both $B \cap \{1, 2, \ldots, k\}$ and $B \cap \{1', 2', \ldots, k'\}$ are nonempty. The *rank* of $d \in \mathcal{P}_k$ (also called the *propagating number*) is

$$\mathsf{rank}(d) = \Big(\text{ the number of propagating blocks in } d\ \Big). \tag{2}$$

For each $k \in \mathbb{Z}_{>0}$, the following are subalgebras of the partition algebra $\mathsf{P}_k(x)$:

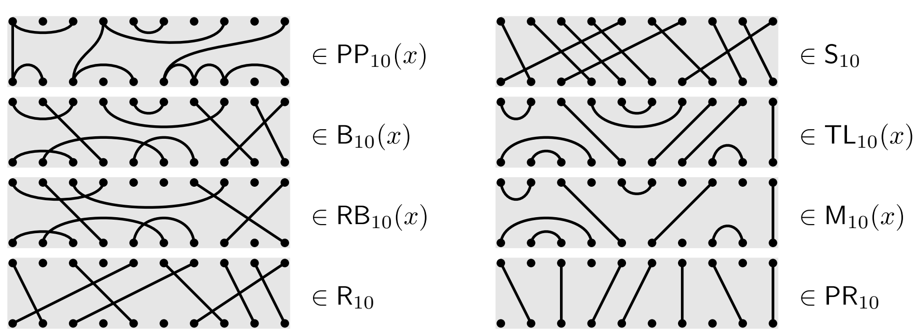
$$\begin{aligned}
\mathbb{C}\mathsf{S}_k &= \mathbb{C}\text{-span}\{\, d \in \mathcal{P}_k \mid \mathsf{rank}(d) = k\},\\
\mathsf{B}_k(x) &= \mathbb{C}\text{-span}\{\, d \in \mathcal{P}_k \mid \text{all blocks of } d \text{ have size } 2\},\\
\mathsf{R}_k &= \mathbb{C}\text{-span}\left\{ d \in \mathcal{P}_k \;\middle|\; \begin{array}{l}\text{all blocks of } d \text{ have at most one vertex in } \{1, \ldots k\}\\ \text{and at most one vertex in } \{1', \ldots k'\}\end{array} \right\},\\
\mathsf{RB}_k(x) &= \mathbb{C}\text{-span}\{\, d \in \mathcal{P}_k \mid \text{all blocks of } d \text{ have size 1 or } 2\}.
\end{aligned}$$

Here, $\mathbb{C}\mathsf{S}_k$ is the group algebra of the symmetric group, $\mathsf{B}_k(x)$ is the Brauer algebra, R_k is the rook monoid algebra [**?**], and $\mathsf{RB}_k(x)$ is the rook-Brauer algebra [**?**], [**?**].

A set partition is *planar* if it can be represented as a diagram without edge crossings inside of the rectangle formed by its vertices. The planar partition algebra [**?**] is $\mathsf{PP}_k(x) = \mathbb{C}\text{-span}\{\, d \in \mathcal{P}_k \mid d \text{ is planar }\}$. The following are the planar subalgebras of $\mathsf{P}_k(x)$:

$$\begin{aligned}
\mathbb{C}\{\mathbf{1}_k\} &= \mathbb{C}\mathsf{S}_k \cap \mathsf{PP}_k(x), & \mathsf{TL}_k(x) &= \mathsf{B}_k(x) \cap \mathsf{PP}_k(x),\\
\mathsf{PR}_k &= \mathsf{R}_k \cap \mathsf{PP}_k(x), & \mathsf{M}_k(x) &= \mathsf{RB}_k(x) \cap \mathsf{PP}_k(x).
\end{aligned}$$

Here, $\mathsf{TL}_k(x)$ is the Temperley-Lieb algebra, PR_k is the planar rook monoid algebra [**?**], and $\mathsf{M}_k(x)$ is the Motzkin algebra [**?**]. The parameter x does not arise when multiplying symmetric group diagrams (as there are never middle blocks to be removed). The parameter is set to be $x = 1$ for the rook monoid algebra and the planar rook monoid algebra. Here are examples from each of these subalgebras:



3 Saxl's Model Representation of the Symmetric Group

An involution $t \in \mathsf{S}_k$ is a permutation such that $t^2 = 1$. In disjoint cycle notation, involutions consist of 2-cycles and fixed points. Let I_k be the set of involutions in S_k and let I_k^f be the involutions in S_k which fix precisely f points. For a fixed involution $t \in \mathsf{I}_k^f$, let $\mathsf{C}(t) \subseteq \mathsf{S}_n$ be the centralizer of t in S_k. If $w \in \mathsf{C}(t)$, then $wtw^{-1} = t$, so w fixes t but possibly permutes the fixed points of t. Let π_f be the linear character of

$\mathsf{C}(t)$ such that $\pi_f(w)$ is the sign of the permutation of w on the fixed points of t. Saxl [**?**] (see also [**?**] or [**?**]) proves the following decomposition of the induced character

$$\varphi^f_{\mathsf{S}_k} := \mathbf{Ind}^{\mathsf{S}_n}_{\mathsf{C}(t)}(\pi_f) = \sum_{\substack{\lambda\vdash k\\ \mathsf{odd}(\lambda)=f}} \chi^\lambda_{\mathsf{S}_k}, \qquad \text{and thus} \qquad \varphi_{\mathsf{S}_k} := \sum_{\ell=0}^{\lfloor k/2\rfloor} \varphi^{k-2\ell}_{\mathsf{S}_k} = \sum_{\lambda\vdash k} \chi^\lambda_{\mathsf{S}_k}, \tag{3}$$

where $\mathsf{odd}(\lambda)$ is the number of odd parts of the partition λ. This result generalizes the classic result (see [**?**, Theorem IV]) for fixed-point-free permutations, i.e., the case where $f = 0$. In this case, there are no fixed points and π_0 is the trivial character of $\mathsf{C}(t)$.

We can then explicitly construct the corresponding induced model. If $w \in \mathsf{S}_k$ and $t \in \mathsf{I}_{n,f}$ then $wtw^{-1} \in \mathsf{I}^f_{\mathsf{S}_k}$ is an involution with the same number f of fixed points as t. However, the relative position of the fixed points are permuted in the map $t \mapsto wtw^{-1}$. Define $\mathsf{sign}(w,t)$ to be the sign of the permutation induced on the fixed points of t under conjugation. That is,

$$\mathsf{sign}(w,t) = (-1)^{|\{\, 1\le i<j\le k \,|\, t(i)=i,\, t(j)=j, \text{ and } w(i)>w(j) \,\}|}. \tag{4}$$

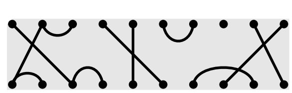
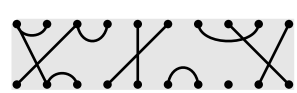
Now, define an action of $w \in \mathsf{S}_k$ on $t \in \mathsf{I}^f_{\mathsf{S}_k}$ by $w \cdot t = \mathsf{sign}(w,t)wtw^{-1}$, which we refer to as *signed conjugation*. Define $\mathsf{M}^f_{\mathsf{S}_k} = \mathbb{C}\text{-span}\{\, t \mid t \in \mathsf{I}^f_{\mathsf{S}_k} \}$, and let S_k act on $\mathsf{M}^f_{\mathsf{S}_k}$ by extending the action linearly. We then prove that $\mathsf{M}^f_{\mathsf{S}_k} \cong \mathbf{Ind}^{\mathsf{S}_k}_{\mathsf{C}(t)}(\mathsf{M}_t)$, and it follows from (3) that

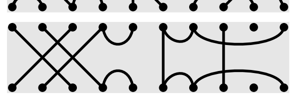
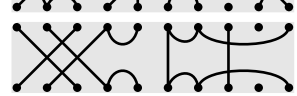
$$\mathsf{M}_{\mathsf{S}_k} = \bigoplus_f \mathsf{M}^f_{\mathsf{S}_k} \cong \bigoplus_{\lambda\vdash n} \mathsf{S}^\lambda_k. \tag{5}$$

Adin, Postnikov, and Roichman [**?**] study a slightly different combinatorial model for S_k. In this work, the sign is computed as $\overline{\mathsf{sign}}(w,t) = (-1)^{|\{\, 1\le i<j\le k \,|\, t(i)=j,\, t(j)=i, \text{ and } w(i)>w(j) \,\}|}$. If we let $\overline{\mathsf{M}}^f_k$ denote the corresponding S_k module, then we are able to prove that $\mathsf{M}^f_{\mathsf{S}_k} \cong \overline{\mathsf{M}}^f_{\mathsf{S}_k} \otimes \mathsf{S}^{(1^k)}_k$, where $\mathsf{S}^{(1^k)}_k$ is the sign representation of S_k.

4 Gelfand Models for Diagram Algebras

Let A_k be any one of the diagrams described in Section 2 with the parameter $x \in \mathbb{C}$ chosen such that A_k is semisimple. Let $\mathcal{A}_k$ be the basis of diagrams which span A_k. For $d \in \mathcal{A}_k$, let $d^T \in \mathcal{A}_k$ be the diagram obtained by reflecting d over its horizontal axis. Note that the map $d \to d^T$ corresponds to exchanging $i \leftrightarrow i'$ for all i. For example,

$d_1 =$  $\Rightarrow$ $d_1^T =$  ,

$d_2 =$  $\Rightarrow$ $d_2^T =$  .

We say that a diagram d is *symmetric* if $d^T = d$, so that d_2 is symmetric and d_1 is not. If we let $(i')' = i$ and let $B' = \{\, b' \,|\, b \in B \,\}$ for a block B of a partition diagram d, then d is symmetric if it satisfies: $B \in d$ if and only if $B' \in d$. If d is a partition diagram, then we say that a block $B \in d$ is a *fixed block* if $B' = B$. In our above examples, d_1 has one fixed block, $\{5, 5'\}$, and d_2 has two fixed blocks, $\{8, 8'\}$ and $\{6, 7, 10, 6', 7', 10'\}$. Note that for $a, b \in \mathcal{A}_k$, $(ab)^T = b^T a^T$, and observe that $(dtd^T)^T = (d^T)^T t^T d^T = dtd^T$, so t is symmetric if and only if dtd^T is symmetric. We say that dtd^T is the *conjugate* of t by d.

Remark 6 *The symmetric diagrams in this paper are the same as the type-B set partitions in [**?**] Sequence A002872 and they are closely related to the type-B set partitions used in [**?**].*

Remark 7 *If we restrict our diagrams to S_k, then d^T equals d^{-1}, diagram conjugation corresponds to usual group conjugation, symmetric diagrams are involutions, and fixed blocks are fixed points.*

For any of our diagram algebras A_k, we let

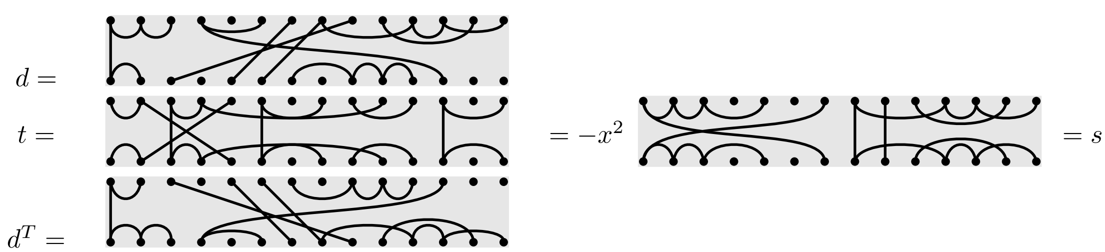
$$\begin{array}{rcl} \mathsf{I}^{r,f}_{\mathsf{A}_k} & = & \{\, d \in \mathcal{A}_k \mid d \text{ is symmetric, } \mathsf{rank}(d) = r \text{, and } d \text{ has } f \text{ fixed blocks} \,\}, \\ \mathsf{I}^{r}_{\mathsf{A}_k} & = & \{\, d \in \mathcal{A}_k \mid d \text{ is symmetric, } \mathsf{rank}(d) = r \,\}, \\ \mathsf{I}_{\mathsf{A}_k} & = & \{\, d \in \mathcal{A}_k \mid d \text{ is symmetric} \,\}, \end{array} \tag{8}$$

If $d \in \mathcal{A}_k$ and $t \in \mathsf{I}^{r,f}_{\mathsf{A}_k}$, then there are two possibilities for the map $t \mapsto d \circ t \circ d^T$. Either $\mathsf{rank}(d \circ t \circ d^T) < \mathsf{rank}(t)$ or $\mathsf{rank}(d \circ t \circ d^T) = \mathsf{rank}(t)$. In the later case, the fixed blocks of t have been permuted, and we let $\mathsf{sign}(d, t)$ be the sign of the permutation of the fixed blocks of t. and for $d \in \mathcal{A}_k$ and $t \in \mathsf{I}^{r,f}_{\mathsf{A}_k}$, we define

$$d \cdot t = \begin{cases} x^{\kappa(d,t)} \mathsf{sign}(d,t)\, d \circ t \circ d^T, & \text{if } \mathsf{rank}(d \circ t \circ d^T) = \mathsf{rank}(t) \\ 0, & \text{if } \mathsf{rank}(d \circ t \circ d^T) < \mathsf{rank}(t) \end{cases} \tag{9}$$

where $\kappa(d, t)$ is the number of blocks removed from the middle row in creating $d \circ t$ as described in (1).

Example 10 (Signed Conjugation) *In the following example, there are two blocks removed in $d \circ t$ yielding x^2. Furthermore, the three fixed blocks of t are permuted as $(B_1, B_2, B_3) \mapsto (B_3, B_2, B_1)$. Hence,* $\mathsf{sign}(d, t) = -1$.



For $0 \le f \le r \le k$, define $\mathsf{M}^{r,f}_{\mathsf{A}_k} = \mathbb{C}\text{-span}\{\, d \mid d \in \mathsf{I}^{r,f}_{\mathsf{A}_k} \,\}$, where $\mathsf{M}^{r,f}_{\mathsf{A}_k} = 0$ if $\mathsf{I}^{r,f}_{\mathsf{A}_k} = \emptyset$, and let

$$\begin{array}{rcl} \mathsf{M}^{r}_{\mathsf{A}_k} & = & \mathbb{C}\text{-span}\{\, d \mid d \in \mathsf{I}^{r}_{\mathsf{A}_k} \,\}, \\ & = & \displaystyle\bigoplus_{f=0}^{r} \mathsf{M}^{r,f}_{\mathsf{A}_k}, \end{array} \quad \text{and} \quad \begin{array}{rcl} \mathsf{M}_{\mathsf{A}_k} & = & \mathbb{C}\text{-span}\{\, d \mid d \in \mathsf{I}_{\mathsf{A}_k} \,\}, \\ & = & \displaystyle\bigoplus_{r=0}^{k} \mathsf{M}^{r}_{\mathsf{A}_k} = \bigoplus_{r=0}^{k} \bigoplus_{f=0}^{r} \mathsf{M}^{r,f}_{\mathsf{A}_k}. \end{array} \tag{11}$$

Then we prove the following:

Proposition 12 *The action defined in* (9) *makes* $\mathsf{M}^{r,f}_{\mathsf{A}_k}$ *an* A_k*-module.*

The main theorem of this paper is the following.

Theorem 13 *For each* $0 \le f \le r \le k$ *chosen such that* $\mathsf{M}^{r,f}_{\mathsf{A}_k} \neq 0$*, we have*

$$\mathsf{M}^{r,f}_{\mathsf{A}_k} \cong \bigoplus_{\lambda \in \Lambda^f_{\mathsf{C}_r}} \mathsf{M}^{\lambda}_{\mathsf{A}_k} \qquad \textit{and thus} \qquad \mathsf{M}_{\mathsf{A}_k} \cong \bigoplus_{\lambda \in \Lambda_{\mathsf{A}_k}} \mathsf{M}^{\lambda}_{\mathsf{A}_k}.$$

Our method of proof of this theorem is to use the Jones basic construction. We have a natural tower of algebras, $\mathsf{A}_0 \subseteq \mathsf{A}_1 \subseteq \mathsf{A}_2 \subseteq \cdots$. where A_{k-1} is embedded as subalgebra of A_k by placing an identity edge to the right of any diagram in A_{k-1}. Let $\mathsf{J}_{k-1} \subseteq \mathsf{A}_k$ be the ideal spanned by the diagrams of A_k having rank $k-1$ *or less*. Then,

$$\mathsf{A}_k \cong \mathsf{J}_{k-1} \oplus \mathsf{C}_k, \tag{14}$$

where C_k is the span of the diagrams of rank exactly equal to k. For us,

$$\begin{array}{ll} \mathsf{C}_k \cong \mathbb{C}\mathsf{S}_k & \text{when } \mathsf{A}_k \text{ is one of the nonplanar algebras } \mathsf{P}_k(x), \mathsf{B}_k(x), \mathsf{RB}_k(x) \text{ or } \mathsf{R}_k, \\ \mathsf{C}_k \cong \mathbb{C}\mathbf{1}_k & \text{when } \mathsf{A}_k \text{ is one of the planar algebras } \mathsf{TL}_k(x), \mathsf{M}_k(x), \text{ or } \mathsf{PR}_k, \end{array} \tag{15}$$

We then are able to lift model representations from $\mathsf{C}_r, 0 \le r \le k$, to a model for A_k.

5 Gelfand Models for Diagram Algebras

We now illustrate some of the combinatorial details that come from applying our model construction to the various diagram algebras.

5.1 The partition algebra $\mathsf{P}_k(x)$

The partition algebra $\mathsf{P}_k(x)$ has dimension equal to the Bell number $B(2k)$ and is semisimple for $x \in \mathbb{C}$ such that $x \notin \{0, 1, \dots, 2k-1\}$ (see [**?**] or [**?**]). When semisimple, its irreducible representations are indexed by partitions in the set $\Lambda_{\mathsf{P}_k} = \{\ \lambda \vdash r \mid 0 \le r \le k\ \}$. Let P^{λ}_k denote the irreducible module indexed by $\lambda \in \Lambda_{\mathsf{P}_k}$.

For each $0 \le \ell \le \lfloor r/2 \rfloor$ there exist symmetric diagrams in $\mathsf{I}^{r,f}_{\mathsf{P}_k}$ of rank r with $f = r - 2\ell$ fixed blocks and ℓ blocks which are transposed (i.e., propagating, nonidentity blocks). The model representation satisfies

$$\mathsf{M}^{r,f}_{\mathsf{P}_k} = \sum_{\substack{\lambda \vdash k \\ \mathsf{odd}(\lambda) = f}} \mathsf{P}_k{}^{\lambda} \qquad \text{and} \qquad \mathsf{M}_{\mathsf{P}_k} = \sum_{r=0}^{k} \sum_{\ell=0}^{\lfloor r/2 \rfloor} \mathsf{M}^{r,r-2\ell}_{\mathsf{P}_k} = \sum_{\lambda \in \Lambda_{\mathsf{P}_k}} \mathsf{P}_k{}^{\lambda}. \tag{16}$$

We show that

$$\dim \mathsf{M}^{r,r-2\ell}_{\mathsf{P}_k} = \left| \mathsf{I}^{r,r-2\ell}_{\mathsf{P}_k} \right| = \sum_{b=r}^{k} S(k,b) \binom{b}{r} \binom{r}{2\ell} (2\ell - 1)!! \tag{17}$$

where $S(k,b)$ is a Stirling number of the second kind. If we let $\mathsf{p}_k = |\mathsf{I}_{\mathsf{P}_k}| = \sum_{r=0}^{k} \sum_{\ell=0}^{\lfloor r/2 \rfloor} |\mathsf{I}^{r,r-2\ell}_{\mathsf{P}_k}| = \dim \mathsf{M}_{\mathsf{P}_k}$ denote the total number of symmetric diagrams in $\mathsf{P}_k(x)$, then p_k is the sum of the degrees of the irreducible $\mathsf{P}_k(x)$-modules (which can be found in [?], [?]). The first few values are

$$\begin{array}{r|ccccccccccc} k & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ \hline \mathsf{p}_k = \dim \mathsf{M}_{\mathsf{P}_k} & 1 & 2 & 7 & 31 & 164 & 999 & 6841 & 51790 & 428131 & 3827967 & 36738144 \end{array}. \tag{18}$$

The sequence p_k is [?] Sequence A002872, which equals the number of type-B set partitions (see Remark 6) and has exponential generating function $e^{(e^{2x}-3)/2+e^x} = \sum_{k=0}^{\infty} \mathsf{p}_k \frac{x^k}{k!}$.

5.2 The Brauer algebra $\mathsf{B}_k(x)$

The Brauer algebra has dimension $\dim \mathsf{B}_k(x) = (2k-1)!!$ and is semisimple for $x \in \mathbb{C}$ chosen to avoid $\{ x \in \mathbb{Z} \,|\, 4-2k \leq x \leq k-2\}$. When $\mathsf{B}_k(x)$ is semisimple, its irreducible modules are indexed by partitions in the set $\Lambda_{\mathsf{B}_k} = \{ \lambda \vdash (k-2r) \mid 0 \leq r \leq \lfloor k/2 \rfloor \}$. Let B_k^λ denote the irreducible $\mathsf{B}_k(x)$ module for $\lambda \in \Lambda_{\mathsf{B}_k}$.

For each $0 \leq c \leq \lfloor k/2 \rfloor$ and each $0 \leq \ell \leq \lfloor (k-2c)/2 \rfloor$ there exist symmetric diagrams in $\mathsf{I}_{\mathsf{B}_k}^{k-2c,k-2c-2\ell}$ of rank $r = k-2c$ with $f = k-2c-2\ell$ fixed blocks. The $\mathsf{B}_k(x)$ model satisfies

$$\mathsf{M}_{\mathsf{B}_k}^{r,f} \cong \bigoplus_{\substack{\lambda \vdash r \\ \mathrm{odd}(\lambda)=f}} \mathsf{B}_k^\lambda \qquad \text{and} \qquad \mathsf{M}_{\mathsf{B}_k} \cong \bigoplus_{c=0}^{\lfloor k/2 \rfloor} \bigoplus_{\ell=0}^{\lfloor (k-2c)/2 \rfloor} \mathsf{M}_{\mathsf{B}_k}^{k-2c,k-2c-2\ell} \cong \bigoplus_{\lambda \in \Lambda_{\mathsf{B}_k}} \mathsf{B}_k^\lambda. \tag{19}$$

We show that

$$\dim \mathsf{M}_{\mathsf{B}_k}^{r,r-2\ell} = \left| \mathsf{I}_{\mathsf{B}_k}^{r,r-2\ell} \right| = \binom{k}{r} (k-r-1)!! \binom{r}{2\ell} (2\ell-1)!!. \tag{20}$$

If we let $\mathsf{b}_k = |\mathsf{I}_{\mathsf{B}_k}| = \sum_{c=0}^{\lfloor k/2 \rfloor} \sum_{\ell=0}^{\lfloor (k-2c)/2 \rfloor} |\mathsf{I}_{\mathsf{B}_k}^{k-2c,k-2c-2\ell}| = \dim \mathsf{M}_{\mathsf{B}_k}$ denote the total number of symmetric diagrams in $\mathsf{B}_k(x)$, then b_k is the sum of the degrees of the irreducible $\mathsf{B}_k(x)$-modules (which can be found in [?]). The first few values of these dimensions are

$$\begin{array}{r|ccccccccccc} k & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ \hline \mathsf{b}_k = \dim \mathsf{M}_{\mathsf{B}_k} & 1 & 1 & 3 & 7 & 25 & 81 & 331 & 1303 & 5937 & 26785 & 133651 \end{array}. \tag{21}$$

The sequence b_k is [?] Sequence A047974 and has exponential generating function $e^{x^2+x} = \sum_{k=0}^{\infty} \mathsf{b}_k \frac{x^k}{k!}$.

5.3 The rook monoid algebra R_k

The rook monoid algebra R_k has dimension $\dim \mathsf{R}_k = \sum_{\ell=0}^{k} \binom{k}{\ell}^2 \ell!$ (see [?], [?], [?]) and is semisimple with irreducible modules labeled by $\Lambda_{\mathsf{R}_k} = \{ \lambda \vdash r \mid 0 \leq r \leq \lfloor k \rfloor \}$. Let R_k^λ denote the irreducible module labeled by $\lambda \in \Lambda_{\mathsf{R}_k}$.

For each $0 \leq r \leq k$ and each $0 \leq \ell \leq \lfloor r/2 \rfloor$ there exist symmetric rook monoid diagrams of rank r and $f = r - 2\ell$ fixed blocks. The R_k model satisfies

$$\mathsf{M}_{\mathsf{R}_k}^{r,f} \cong \bigoplus_{\substack{\lambda \vdash r \\ \mathrm{odd}(\lambda)=f}} \mathsf{R}_k^\lambda \qquad \text{and} \qquad \mathsf{M}_{\mathsf{R}_k} \cong \bigoplus_{r=0}^{k} \bigoplus_{\ell=0}^{\lfloor r/2 \rfloor} \mathsf{M}_{\mathsf{R}_k}^{r,r-2\ell} \cong \bigoplus_{\lambda \in \Lambda_{\mathsf{R}_k}} \mathsf{R}_k^\lambda. \tag{22}$$

We show that

$$\dim \mathsf{M}_{\mathsf{R}_k}^{r,r-2\ell} = \left| \mathsf{I}_{\mathsf{R}_k}^{r,r-2\ell} \right| = \binom{k}{r} \binom{r}{2\ell} (2\ell-1)!!, \tag{23}$$

If we let $\mathsf{r}_k = |\mathsf{I}_{\mathsf{R}_k}| = \sum_{r=0}^{k} \sum_{\ell=0}^{\lfloor r/2 \rfloor} |\mathsf{I}_{\mathsf{R}_k}^{r,r-2\ell}| = \dim \mathsf{M}_{\mathsf{R}_k}$ denote the total number of symmetric diagrams in R_k, then r_k is sum of the degrees of the irreducible R_k-modules (which can be found in [?], [?]). The first few values of these dimensions are

k	0	1	2	3	4	5	6	7	8	9	10
$\dim \mathsf{M}_{\mathsf{R}_k}$	1	2	5	14	43	142	499	1850	7193	29186	123109

(24)

The sequence r_k gives the number of "self-inverse partial permutations" and is [?] Sequence A005425. Furthermore, r_k is related to the number of involutions s_k in the symmetric group by the binomial transform $\mathsf{r}_k = \sum_{i=0}^{k} \binom{k}{i} \mathsf{s}_i$ and thus has exponential generating function $e^{x^2/2+2x} = \sum_{k=0}^{\infty} \mathsf{r}_k \frac{x^k}{k!}$.

5.4 The rook-Brauer algebra $\mathsf{RB}_k(x)$

The rook-Brauer algebra $\mathsf{RB}_k(x)$ (see [?] or [?]) has dimension $\sum_{\ell=0}^{k} \binom{2k}{2\ell}(2\ell-1)!!$ and is semisimple for all but finitely many $x \in \mathbb{C}$. When semisimple, its irreducible representations are indexed by partitions in the set $\Lambda_{\mathsf{RB}_k} = \{ \lambda \vdash r \mid 0 \le r \le \lfloor k \rfloor \}$. Let RB_k^{λ} denote the irreducible module indexed by $\lambda \in \Lambda_{\mathsf{RB}_k}$.

For each $0 \le r \le k$ and each $0 \le \ell \le \lfloor r/2 \rfloor$ there exist symmetric rook monoid diagrams of rank r and $f = r - 2\ell$ fixed blocks. The $\mathsf{RB}_k(x)$ models satisfy

$$\mathsf{M}_{\mathsf{RB}_k}^{r,f} \cong \bigoplus_{\substack{\lambda \vdash r \\ \mathsf{odd}(\lambda)=f}} \mathsf{RB}_k^{\lambda} \qquad \text{and} \qquad \mathsf{M}_{\mathsf{RB}_k} \cong \bigoplus_{r=0}^{k} \bigoplus_{\ell=0}^{\lfloor r/2 \rfloor} \mathsf{M}_{\mathsf{RB}_k}^{r,r-2\ell} \cong \bigoplus_{\lambda \in \Lambda_{\mathsf{RB}_k}} \mathsf{RB}_k^{\lambda}. \tag{25}$$

We show that

$$\dim \mathsf{M}_{\mathsf{RB}_k}^{r,r-2\ell} = \left| \mathsf{I}_{\mathsf{RB}_k}^{r,r-2\ell} \right| = \sum_{c=0}^{\lfloor (k-r)/2 \rfloor} \binom{k}{r} \binom{k-r}{2c} (2c-1)!! \binom{r}{2\ell} (2\ell-1)!!. \tag{26}$$

If we let $\mathsf{rb}_k = |\mathsf{I}_{\mathsf{RB}_k}| = \sum_{r=0}^{k} \sum_{\ell=0}^{\lfloor r/2 \rfloor} |\mathsf{I}_{\mathsf{RB}_k}^{r,r-2\ell}| = \dim \mathsf{M}_{\mathsf{RB}_k}$ denote the total number of symmetric diagrams in $\mathsf{RB}_k(x)$, then rb_k is the sum of the degrees of the irreducible $\mathsf{RB}_k(x)$-modules (these dimensions can be found in [?] or [?]). The first few values of these dimensions are

k	0	1	2	3	4	5	6	7	8	9	10
$\mathsf{rb}_k = \dim \mathsf{M}_{\mathsf{RB}_k}$	1	2	6	20	76	312	1384	6512	32400	168992	921184

(27)

The sequence rb_k is [?] Sequence A000898 and it is related to the number of symmetric diagrams b_k in the Brauer algebra (21) by the binomial transform $\mathsf{rb}_k = \sum_{i=0}^{k} \binom{k}{i} \mathsf{b}_i$ and thus has exponential generating function $e^{x^2+2x} = \sum_{k=0}^{\infty} \mathsf{rb}_k \frac{x^k}{k!}$.

5.5 The Temperley-Lieb algebra $\mathsf{TL}_k(x)$

The Temperley-Lieb algebra $\mathsf{TL}_k(x)$ has dimension equal to the Catalan number $C_k = \frac{1}{k+1}\binom{2k}{k}$ and is semisimple for $x \in \mathbb{C}$ chosen such that x is not the root of the Chebyshev polynomial $U_k(x/2)$ (see [?] or [?]). When semisimple, its irreducible modules are indexed by the following set of integers $\Lambda_{\mathsf{TL}_k} = \{ k - 2\ell \mid 0 \le \ell \le \lfloor k/2 \rfloor \}$. We let $\mathsf{TL}_k^{(k-2\ell)}$ denote the irreducible module labeled by $(k-2\ell) \in \Lambda_{\mathsf{TL}_k}$

For each $0 \le \ell \le \lfloor k/2 \rfloor$, there exist symmetric Temperley-Lieb diagrams of rank $r = k - 2\ell$ and $f = k - 2\ell$ fixed points. The $\mathsf{TL}_k(x)$ model satisfies

$$\mathsf{M}_{\mathsf{TL}_k}^{(k-2\ell)} \cong \mathsf{TL}_k^{(k-2\ell)} \qquad \text{and} \qquad \mathsf{M}_{\mathsf{TL}_k} \cong \bigoplus_{\ell=0}^{\lfloor k/2 \rfloor} \mathsf{M}_{\mathsf{TL}_k}^{(k-2\ell)} \cong \bigoplus_{(k-2\ell)\in\Lambda_{\mathsf{TL}_k}} \mathsf{TL}_k^{(k-2\ell)}. \tag{28}$$

The number of symmetric Temperley-Lieb diagrams of rank r with $r = f$ fixed points is given by

$$\dim \mathsf{M}_{\mathsf{TL}_k}^{r,f} = \left|\mathsf{I}_{\mathsf{TL}_k}^{k-2\ell}\right| = \left\{ \begin{matrix} k \\ \ell \end{matrix} \right\} := \binom{k}{\ell} - \binom{k}{\ell - 1} \tag{29}$$

If we let $\mathsf{tl}_k = |\mathsf{I}_{\mathsf{TL}_k}| = \sum_{\ell=0}^{\lfloor k/2 \rfloor} |\mathsf{I}_{\mathsf{TL}_k}^{k-2\ell}| = \dim \mathsf{M}_{\mathsf{TL}_k}$ denote the total number of symmetric diagrams in $\mathsf{TL}_k(x)$, then tl_k is the sum of the degrees of the irreducible $\mathsf{TL}_k(x)$-modules. We give a bijection between the symmetric Temperley-Lieb diagrams $\mathsf{I}_{\mathsf{TL}_k}$ and subsets of $\{1, 2, \ldots, k\}$ of size $\lfloor k/2 \rfloor$ and thus $\mathsf{tl}_k = \binom{k}{\lfloor k/2 \rfloor}$ (the kth central binomial coefficient), which is [**?**] Sequence A000984.

5.6 The Motzkin algebra $\mathsf{M}_k(x)$

The Motzkin algebra $\mathsf{M}_k(x)$ has dimension equal to the Motzkin number M_{2k} (see [**?**]) and is semisimple for $x \in \mathbb{C}$ chosen such that x is not the root of the Chebyshev polynomial $U_k((x-1)/2)$. When semisimple, its the irreducible modules are indexed by $\Lambda_{\mathsf{M}_k} = \{0, 1, \ldots, k\}$. We let $\mathsf{M}_k^{(r)}$ denote the irreducible module labeled by $r \in \Lambda_{\mathsf{M}_k}$.

For each $0 \le r \le k$ there exist symmetric Motzkin diagrams having rank r and $f = r$ fixed blocks. The $\mathsf{M}_k(x)$ models satisfy

$$\mathsf{M}_{\mathsf{M}_k}^{r} \cong \mathsf{M}_k^{(r)} \qquad \text{and} \qquad \mathsf{M}_{\mathsf{M}_k} \cong \bigoplus_{r=0}^{k} \mathsf{M}_{\mathsf{M}_k}^{r} \cong \bigoplus_{r\in\Lambda_{\mathsf{M}_k}} \mathsf{M}_k^{(r)}. \tag{30}$$

We show that

$$\dim \mathsf{M}_{\mathsf{M}_k}^{r} = \left|\mathsf{I}_{\mathsf{M}_k}^{r}\right| = \sum_{c=0}^{\lfloor (k-r)/2 \rfloor} \binom{k}{r+2c} \left\{ \begin{matrix} r+2c \\ c \end{matrix} \right\}. \tag{31}$$

If we let $\mathsf{m}_k = |\mathsf{I}_{\mathsf{M}_k}| = \sum_{r=0}^{k} |\mathsf{I}_{\mathsf{M}_k}^{r}| = \dim \mathsf{M}_{\mathsf{M}_k}$ denote the total number of symmetric diagrams in $\mathsf{M}_k(x)$, then m_k is the degree of φ_{M_k} and is the sum of the degrees of the irreducible $\mathsf{M}_k(x)$-modules. The first few values of these dimensions are

k	0	1	2	3	4	5	6	7	8	9	10
$\mathsf{m}_k = \dim \mathsf{M}_{\mathsf{M}_k}$	1	2	5	13	35	96	267	750	2123	6046	17303

(32)

The sequence m_k is [**?**] Sequence A005773 and it is related to the number of symmetric diagrams tl_k in the Temperley-Lieb algebra by the binomial transform $\mathsf{m}_k = \sum_{i=0}^{k} \binom{k}{i} \mathsf{tl}_i$ and thus has exponential generating function $e^x(I_0(2x) + I_1(2x)) = \sum_{k=0}^{\infty} \mathsf{m}_k \frac{x^k}{k!}$.

5.7 The planar rook monoid algebra PR_k

The planar rook monoid algebra PR_k has dimension $\binom{2k}{k}$ and is semisimple with irreducible modules labeled by $\Lambda_{\mathsf{PR}_k} = \{0, 1, \ldots, k\}$. We let $\mathsf{PR}_k^{(r)}$ denote the irreducible PR_k-module labeled by $r \in \Lambda_{\mathsf{PR}_k}$

For each $0 \leq r \leq k$ there exist $\binom{k}{r}$ symmetric planar rook monoid diagrams having rank r and $f = r$ fixed blocks. The PR_k model satisfies

$$\mathsf{M}^r_{\mathsf{PR}_k} \cong \mathsf{PR}_k^{(r)} \qquad \text{and} \qquad \mathsf{M}_{\mathsf{PR}_k} \cong \bigoplus_{r=0}^{k} \mathsf{M}^r_{\mathsf{PR}_k} \cong \bigoplus_{r \in \Lambda_{\mathsf{PR}_k}} \mathsf{PR}_k^{(r)}. \tag{33}$$

The irreducible modules $\mathsf{PR}_k^{(r)}$ are constructed in [**?**] on a basis of r-subsets of $\{1, 2, \ldots, k\}$. The action of PR_k on subsets is exactly the same as our conjugation action on symmetric diagrams. If we let $\mathsf{pr}_k = |\mathsf{I}_{PR_k}| = \sum_{r=0}^{k} |\mathsf{I}^r_{PR_k}| = \dim \mathsf{M}_{\mathsf{PR}_k}$ denote the total number of symmetric diagrams in PR_k, then pr_k is the number of subsets of $\{1, 2, \ldots, k\}$, so $\mathsf{pr}_k = \dim \mathsf{M}_{\mathsf{PR}_k} = 2^k$.

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References