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The Robinson–Schensted Correspondence and A_2-webs

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Abstract. The A_2-spider category encodes the representation theory of the sl_3 quantum group. Kuperberg (1996) introduced a combinatorial version of this category, wherein morphisms are represented by planar graphs called *webs* and the subset of *reduced webs* forms bases for morphism spaces. A great deal of recent interest has focused on the combinatorics of invariant webs for tensors powers of V^+, the standard representation of the quantum group. In particular, the invariant webs for the $3n$th tensor power of V^+ correspond bijectively to $[n, n, n]$ standard Young tableaux. Kuperberg originally defined this map in terms of a graphical algorithm, and subsequent papers of Khovanov–Kuperberg (1999) and Tymoczko (2012) introduce algorithms for computing the inverse. The main result of this paper is a redefinition of Kuperberg's map through the representation theory of the symmetric group. In the classical limit, the space of invariant webs carries a symmetric group action. We use this structure in conjunction with Vogan's generalized tau-invariant and Kazhdan–Lusztig theory to show that Kuperberg's map is a direct analogue of the Robinson–Schensted correspondence.

Résumé. La catégorie d'araignée A_2 encode la théorie des représentations du groupe quantique $U_q(sl_3)$. Kuperberg (1996) a introduit une version combinatoire de cette catégorie, dans laquelle les morphismes sont reprśentés par des graphes planaires appelés *toiles* dont le sous-ensemble de *toiles réduites* constitue des bases pour les espaces de morphismes. Beaucoup d'interêt a été fixé recemment sur le combinatoire des toiles invariantes pour les puissances tensorielles de V^+, la représentation standarde de $U_q(sl_3)$. En particulier, les toiles invariantes pour $(V^+)^{3n}$ sont en corresponance bijective au tableaux de Young standards $[n, n, n]$. L'application original de Kuperberg avait une définition en termes d'un algorithme graphique, puis des articles de Khovanov–Kuperberg (1999) et Tymoczko (2012) présentent des algorithmes pour la computation d l'inverse. Le résultat principal de cette article est une redéfinition de l'application de Kuperberg à travers la théorie de représentations du groupe symétrique. Dans la limite classique, l'éspace des toiles invariantes porte une action de S_{3n}. On emploie cette structure de concert avec l'invariant tau généralisé de Vogan et la théorie Kazhdan–Lusztig pour montrer que l'application de Kuperberg est un analogue direct de la correspondance Robinson–Schensted.

Keywords: Robinson–Schensted, Web basis, Kazhdan–Lusztig theory, Young tableau

1 Introduction

The A_2-spider is a category encoding the representation theory of $\mathcal{U}_q(\mathfrak{sl}_3)$, the quantum enveloping algebra of the $\mathfrak{sl}_3$ Lie algebra. The objects in the category are tensor products of V^+ and V^-, the standard and dual representations of the quantum group, while the morphisms are intertwining maps. Kuperberg (1996) defined a diagramatic construction of the A_2-spider in which morphism are represented by planar graphs called *webs*, with the subset of *reduced webs* forming bases for each morphism space.

In order to prove that reduced webs span each morphism space, Kuperberg studied quantum invariants, which may be viewed as morphisms from the trivial representation to other representations. Classical results require that the invariant space for a tensor product of copies of V^+ and V^- has dimension equal to the number of *dominant lattice paths* satisfying certain conditions; Kuperberg developed an explicit graphical algorithm carrying reduced webs to dominant lattice paths and showed that it was bijective. In a subsequent paper Khovanov and Kuperberg (1999) introduced a method to compute the inverse via a recursive growth algorithm. Recent combinatorial interest in Kuperberg's map has focused on the case of invariants for tensor powers of V^+ Petersen et al. (2009); Tymoczko (2012). For $(V^+)^{\otimes 3n}$, the map may be interpreted as a bijection between webs on $3n$ source vertices and standard Young tableaux on the shape $[n, n, n]$.

In this paper, we reinterpret Kuperberg's bijection in terms of the representation theory of the symmetric group. A tensor power of a quantum group representation carries a Hecke algebra action which in the classical limit reduces to a symmetric group action by permutation of tensor factors. The subspace of invariants forms a subrepresentation.

Vogan introduced the generalized τ-invariant to study infinite dimensional representations of semisimple Lie algebras Vogan (1979). Generalized τ-invariants are closely related to the combinatorics of standard Young tableaux (Section 2). The generalized τ-invariant gives a nonalgorithmic way of defining the Robinson–Schensted correspondence between symmetric group elements and same shape pairs of standard Young tableaux (Section 3). In Section 4, we discuss an *in situ* version of the Robinson–Schensted algorithm for parameterizing Kazhdan–Lusztig left cell basis elements by Young tableaux in terms of the symmetric group action on the left cell representation. In Sections 6 and 7, we discuss the symmetric group action on webs and Kuperberg's bijection between reduced webs and standard tableaux. Our main result is in Section 8, where we show that Kuperberg's map can be defined in terms of the Robinson–Schensted algorithm for Kazhdan–Lusztig left cells.

2 Generalized τ-invariants for Tableaux

In this and subsequent sections, we will rely extensively on standard results from the combinatorics of tableaux and the symmetric group. Björner and Brenti (2005) provide an excellent exposition of this material.

Recall that a *Young diagram* is a collection of finitely many boxes arranged in left justified rows so that no row has more boxes than the rows above it. Young diagrams with n boxes correspond naturally to partitions of n by treating the length of each row as an element of a partition. A *standard Young tableau* on a Young diagram with n boxes is a labeling of the boxes with the numbers $1, 2, \ldots, n$ in such a way that the label in each box is less than the labels in the boxes immediately below and immediately to the right, with each label appearing exactly once.

We indicate the set of all standard Young tableaux on n boxes by $\mathscr{T}_n$. Let $s_i \in S_n$ be the simple transposition that exchanges i and $i+1$. Given $Y \in \mathscr{T}_n$, $\tau(Y)$ is a subset of the simple transpositions in S_n, where $s_i \in \tau(Y)$ when $i+1$ is below the row of i in Y. We refer to $\tau(Y)$ as the *τ-invariant* of Y. (Most sources call $\tau(Y)$ the descent set of Y, but we choose our terminology to be consistent with Vogan (1979).) If s_i, s_j are adjacent simple transpositions in S_n, then $D^{\mathrm{YT}}_{i,j}$ is defined as the set of all $Y \in \mathscr{T}_n$

such that $s_i \in \tau(Y)$ and $s_j \notin \tau(Y)$. Let $s_i \cdot Y$ be the (not necessarily standard) tableau obtained from Y by exchanging i and $i+1$.

Lemma 1 *Let s_i, s_j be adjacent simple transpositions in S_n. Given $Y \in D^{\mathrm{YT}}_{i,j}$, exactly one of $s_i \cdot Y$ and $s_j \cdot Y$, denoted $f^{\mathrm{YT}}_{i,j}(Y)$, is a standard tableau in $D^{\mathrm{YT}}_{j,i}$; $f^{\mathrm{YT}}_{i,j} : D^{\mathrm{YT}}_{i,j} \to D^{\mathrm{YT}}_{j,i}$ is a bijection with inverse $f^{\mathrm{YT}}_{j,i} : D^{\mathrm{YT}}_{j,i} \to D^{\mathrm{YT}}_{i,j}$.*

Definition 1 *Let Y and Y' be elements of $\mathscr{T}_n$. If $\tau(Y) = \tau(Y')$, then Y and Y' are* equivalent to order 0*, denoted $Y \underset{0}{\approx} Y'$. We say that $Y \underset{n}{\approx} Y'$ (Y and Y' are equivalent to order n) if $Y \underset{n-1}{\approx} Y'$ and $f^{\mathrm{YT}}_{i,j}(Y) \underset{n-1}{\approx} f^{\mathrm{YT}}_{i,j}(Y')$ whenever Y and Y' are in $D^{\mathrm{YT}}_{i,j}$. If $Y \underset{n}{\approx} Y'$ for all nonnegative integers n, then Y and Y' have the same* generalized τ-invariant*. ($\tau_g(Y) = \tau_g(Y')$).*

Theorem 1 (Vogan) *If $Y, Y' \in \mathscr{T}_n$ and $\tau_g(Y) = \tau_g(Y')$, then $Y = Y'$.*

In other words, a standard Young tableau on n boxes is completely determined by its generalized τ-invariant.

3 Generalized τ-invariants and the Robinson–Schensted Correspondence

We can define τ-invariants and generalized τ-invariants for elements of the symmetric group. As we will see in a moment, these constructions are closely related to the previous definitions for Young tableaux.

Take as a generating set for S_n the simple transpositions $s_1, s_2, \ldots$ Given $x, y \in S_n$, let $x \leq y$ if some minimal length expression for x in terms of generators is a subword of some minimal expression for y. This defines the *Bruhat order* on S_n. We define the τ-invariant for $x \in S_n$ by letting $\tau(x)$ be the set of simple transpositions such that $s_i \cdot x < x$.

The set $\tau(x)$ is closely related to the one line notation for x. Recall that the one line notation for $x \in S_n$ is a permutation of the integers $1, 2, \ldots n$; if x has one line notation $x_1 x_2 x_3 \cdots$ then x sends 1 to x_1, 2 to x_2, etc. It is a standard fact from the combinatorics of the symmetric group that $s_i \in \tau(x)$ if and only if i and $i+1$ appear out of order in the one line notation for x.

If s_i and s_j are adjacent simple transpositions, let $D^{S_n}_{i,j}$ be the set of all $x \in S_n$ such that $s_i \in \tau(x)$, $s_j \notin \tau(x)$. One can prove the next lemma by thinking about the one line notation for x.

Lemma 2 *If $x \in D^{S_n}_{i,j}$, then exactly one of $s_i \cdot x$, $s_j \cdot x$ is an element of $D^{S_n}_{j,i}$. Denote this element of $D^{S_n}_{j,i}$ by $f_{i,j}(x)$. This defines a bijection $f^{S_n}_{i,j} : D^{S_n}_{i,j} \to D^{S_n}_{j,i}$ with inverse $f^{S_n}_{j,i} : D^{S_n}_{j,i} \to D^{S_n}_{i,j}$.*

This allows us to define a generalized τ-invariant for symmetric group elements by making the appropriate substitutions in Definition 1, i.e.,

Definition 2 *Let x and y be elements of S_n. If $\tau(x) = \tau(y)$, then x and y are* equivalent to order 0*, denoted $x \underset{0}{\approx} y$. We say that $x \underset{n}{\approx} y$ (x and y are equivalent to order n) if $x \underset{n-1}{\approx} y$ and $f^{S_n}_{i,j}(x) \underset{n-1}{\approx} f^{S_n}_{i,j}(y)$ whenever x and y are in $D^{\mathrm{YT}}_{i,j}$. If $x \underset{n}{\approx} y$ for all nonnegative integers n, then x and y have the same* generalized τ-invariant*. ($\tau_g(x) = \tau_g(y)$).*

Recall that the *Robinson-Schensted correspondence* gives a bijection between elements of S_n and the set of same shape ordered pairs of standard Young tableaux with n boxes. (We do not explain the algorithm here, but descriptions are available from many sources.) Given $w \in S_n$, let $P(w)$ and $Q(w)$ denote respectively the left and right tableaux in the pair corresponding to w. One obtains the following by thinking carefully about the Robinson–Schensted algorithm in terms of one line notation.

Lemma 3 *Given $x \in S_n$, $s_i \in \tau(x)$ if and only $s_i \in \tau(P(x))$.*

One can use *dual Knuth relations* to prove the next lemma: (See (Björner and Brenti, 2005, Section 6.4)

Lemma 4 *Given $x \in D^{S_n}_{i,j}$, $P(f^{S_n}_{i,j}(x)) = f^{\mathrm{YT}}_{i,j}(P(x))$ and $Q(f^{S_n}_{i,j}(x)) = Q(x)$.*

Combining the lemmas with Theorem 1 yields:

Theorem 2 *Given $x, y \in S_n$, $\tau_g(x) = \tau_g(y)$ if and only if $\tau_g(P(x)) = \tau_g(P(y))$.*

The definition of the generalized τ-invariant we use here differs from the one given in Vogan (1979) and discussed in Kazhdan and Lusztig (1979); Vogan defines a *right* generalized τ-invariant by using the right action of the symmetric group on itself. Our version is the equivalent obtained by using the *left* action. Vogan's right generalized τ-invariant is given in terms of our left version by $\tau_g(x^{-1})$. Left and right generalized τ-invariants give a nonalgorithmic means of defining the Robinson–Schensted correspondence by comparing generalized τ-invariants of tableaux and permutations in the natural way:

Theorem 3 (The Robinson–Schensted Correspondence) *Given $x \in S_n$, $P(x)$ is the unique element of $\mathscr{T}_n$ such that $\tau_g(P(x)) = \tau_g(x)$, while $Q(x)$ is the unique tableau in $\mathscr{T}_n$ such that $\tau_g(Q(x)) = \tau_g(x^{-1})$.*

4 Generalized τ-invariants and Kazhdan–Lusztig Theory

We now recall some facts from Kazhdan and Lusztig (1979).

Definition 3 *Let A be the ring $\mathbb{Z}[v^{1/2}, v^{-1/2}]$. The* Hecke Algebra *$\mathscr{H}_n$ of the symmetric group is the associative A-algebra with generators $T_{s_1}, T_{s_2}, \ldots, T_{s_{n-1}}$ and relations*

$$T_{s_i}T_{s_j} = T_{s_j}T_{s_i} \text{ if } |i-j| > 1 \tag{1}$$

$$T_{s_i}T_{s_j}T_{s_i} = T_{s_j}T_{s_i}T_{s_j} \text{ if } |i-j| = 1 \tag{2}$$

$$(T_{s_i}+1)(T_{s_i}-v) = 0. \tag{3}$$

In addition, given $w \in S_n$, let

$$T_w = T_{s_{i_1}}T_{s_{i_2}} \ldots, T_{s_{i_k}}$$

where

$$s_{i_1}s_{i_2} \ldots, s_{i_k}$$

is some reduced expression for w in terms of the simple transpositions $\{s_i\}$.

Note that $\mathscr{H}_n$ reduces to the group algebra of S_n over $\mathbb{Z}$ when v is set to 1.

The Hecke algebra has a remarkable basis defined in Kazhdan and Lusztig (1979), whose basis elements are also parameterized by elements of the symmetric group. We denote the basis element parameterized by $w \in S_n$ as C_w. The *Kazhdan–Lusztig basis* is encoded in the *Kazhdan–Lusztig graph*, whose vertices are labeled with elements of S_n, where the edge between the vertices labeled with x and y has multiplicity indicated by $\mu(x,y)$. The left action of a generator T_{s_i} on $\mathscr{H}_n$ in terms of the basis $\{C_w\}_{w\in S_n}$ is given as follows:

$$T_{s_i}C_w = \begin{cases} -C_w & \text{if } s_i \in \tau(w), \\ \\ vC_w + v^{1/2} \displaystyle\sum_{\substack{y\in S_n \\ s_i\in\tau(y)}} \mu(w,y)C_y & \text{otherwise.} \end{cases} \tag{4}$$

Treating it as a left $\mathscr{H}_n$-module, we wish to decompose $\mathscr{H}_n$ into irreducible subquotients. For this, we need the notion of a *left cell*.

Definition 4 *Define a binary relation $\preceq_L$ on S_n by letting $x \preceq_L x$ and $x \preceq_L y$ whenever $\mu(x,y) \neq 0$ and $\tau(x) \not\subset \tau(y)$. (This is equivalent to requiring that C_x appears as a summand in $T_{s_i}C_y$ for some i.) Extend $\preceq_L$ to a preorder by imposing transitivity. We refer to $\preceq_L$ as the* left preorder *on S_n.*

Definition 5 *Define an equivalence relation $\sim_L$ on S_n by letting $x \sim_L y$ if $x \preceq_L y$ and $y \preceq_L x$. The equivalence classes under $\sim_L$ are called* left cells*; $\preceq_L$ descends to a partial order on left cells.*

Left cells lead us naturally to the definition of *left cell modules* and *left cell representations*. Let $\mathrm{Cell}(S_n)$ denote the set of left cells in S_n, ordered by $\preceq_L$. Given $\mathcal{C} \in \mathrm{Cell}(S_n)$, define

$$\mathrm{span}_A\,\mathcal{C} = \mathrm{span}_A\{C_w | w \in \mathcal{C}\}.$$

Definition 6 *Given $\mathcal{C} \in \mathrm{Cell}(S_n)$, define the* left cell module *for $\mathcal{C}$ by*

$$\mathrm{KL}_{\mathcal{C}} = \Bigg(\bigoplus_{\substack{\mathcal{C}_i\in\mathrm{Cell}(S_n) \\ \mathcal{C}_i\preceq_L\mathcal{C}}} \mathrm{span}_A\,\mathcal{C}_i \Bigg) \Bigg/ \Bigg(\bigoplus_{\substack{\mathcal{C}_i\in\mathrm{Cell}(S_n) \\ \mathcal{C}_i\prec_L\mathcal{C}}} \mathrm{span}_A\,\mathcal{C}_i \Bigg).$$

If we set $v = 1$ and extend scalars to $\mathbb{C}$, then $\mathrm{KL}_{\mathcal{C}}$ is referred to as the left cell representation *corresponding to the cell $\mathcal{C}$.*

For a left cell representation $\mathrm{KL}_{\mathcal{C}}$, Equation 4 becomes

$$T_{s_i}C_w = \begin{cases} -C_w & \text{if } s_i \in \tau(w), \\ \\ vC_w + v^{1/2} \displaystyle\sum_{\substack{y\in \mathcal{C} \\ s_i\in\tau(y)}} \mu(w,y)C_y & \text{otherwise.} \end{cases} \tag{5}$$

We refer to a Kazhdan-Lusztig graph restricted to a left cell as a *left cell graph*. Let $\widehat{S}_n$ indicate the set of irreducible representations of S_n over the complex numbers. The following parameterization is a standard result from algebraic combinatorics:

Theorem 4 *The elements of $\widehat{S}_n$ are parameterized by partitions of n. In particular, let $\mathbf{p} = [p_1, p_2, \ldots]$ be a partition of n and $\mathbf{t} = [t_1, t_2, \ldots]$ its transpose partition. Then, $\pi_{\mathbf{p}}$ is the unique element of $\widehat{S}_n$ whose restriction to $\Pi_{p_i\in\mathbf{p}}S_{p_i}$ contains a copy of the trivial representation and whose restriction to $\Pi_{t_i\in\mathbf{t}}S_{t_i}$ contains a copy of the sign representation.*

Note that we will often refer to the element of $\widehat{S}_n$ corresponding to some Young diagram, obtained by taking the diagram to its corresponding partition. This parameterization of $\widehat{S}_n$ is closely related to the decomposition of S_n into left cells. As in Section 3, let $P(w)$ and $Q(w)$ denote respectively the left and right tableaux in the pair corresponding to $w \in S_n$.

Theorem 5 *For each left cell $\mathcal{C}$ of S_n, there exists some standard Young tableau $Q_\mathcal{C}$ with n boxes such that*

$$\mathcal{C} = \{w \in S_n | Q(w) = Q_\mathcal{C}\}.$$

Furthermore, $\mathrm{KL}_\mathcal{C}$ is isomorphic as a representation over $\mathbb{C}$ to the irreducible representation parameterized by the shape of $Q_\mathcal{C}$.

With this description of left cells in hand, suppose that Q and Q' are n-box standard Young tableaux with the same shape and let $\mathcal{C}_Q$ and $\mathcal{C}_{Q'}$ be the cells obtained by fixing these as right tableaux. We define a bijection $\phi_{Q,Q'} : \mathcal{C}_Q \to \mathcal{C}_{Q'}$ by

$$(P, Q) \mapsto (P, Q').$$

Theorem 6 *The map $\phi_{Q,Q'}$ is an isomorphism of graphs and preserves τ-invariant data.*

Thus, we can refer to the Kazhdan–Lusztig left cell basis for some element of $\widehat{S}_n$ without specifying a particular left cell; elements of the basis are naturally parameterized by their left tableaux.

5 An *in situ* Robinson–Schensted Algorithm for Left Cells

Suppose that we are given some element of $\widehat{S}_n$ with its associated Kazhdan–Lusztig left cell basis but without the tableaux attached to basis elements. Is it possible to compute these tableaux in terms of the structure of the (based) representation? In the remainder of this section, we will develop an appropriate definition of the generalized τ-invariant that allows us to do this. We will subsequently use this structure to redefine Kuperberg's map as an analogue of the Robinson–Schensted correspondence.

Observe that if s_i is not in $\tau(x)$, then $C_{s_i x}$ appears with multiplicity one in $T_{s_i} C_x$ (Kazhdan and Lusztig, 1979, p. 171, equation 2.3.a). Since $s_i \notin \tau(x)$ or $s_i \notin \tau(s_i x)$, we have the following lemma:

Lemma 5 *Given any $y, s_i \in S_n$, y and $s_i y$ are connected by an edge of multiplicity one.*

Such an edge is often referred to as a *Bruhat edge*.

Lemma 6 *Let $x \in S_n$ such that $x \in D^{S_n}_{i,j}$. Then, x is connected to $y = f^{S_n}_{i,j}(x)$ by an edge of multiplicity 1; furthermore, if $z \neq y$ is any other element of $D^{S_n}_{j,i}$, then $\mu(x, z) = 0$.*

Now, let $\mathcal{C} \subset S_n$ be some left cell. Then, $\mathrm{KL}_\mathcal{C}$ is the associated left cell representation, with basis $B(\mathrm{KL}_\mathcal{C}) = \{C_w | w \in \mathcal{C}\}$. We wish to define a τ-invariant for each basis element by using the S_n-action on $\mathrm{KL}_\mathcal{C}$ (Equation 5).

Definition 7 *Given some basis element $C \in B(\mathrm{KL}_\mathcal{C})$, let $\tau(C) = \{s_i \in S_n | T_{s_i} \cdot C = -C\}$. Given adjacent simple transpositions s_i, s_j, let $D^{\mathrm{KL}_\mathcal{C}}_{i,j}$ be the set of basis elements C in $\mathrm{KL}_\mathcal{C}$ such that $s_i \in \tau(C)$, $s_j \notin \tau(C)$.*

Observation 1 *Notice that $\tau(C_w) = \tau(w)$ as desired.*

Lemma 6 gives rise to the following definition:

Definition 8 *Given $C \in D_{i,j}^{\mathrm{KL}_{\mathcal{C}}}$, let $f_{i,j}^{\mathrm{KL}_{\mathcal{C}}}(C)$ be the unique basis element $C' \in D_{j,i}^{\mathrm{KL}_{\mathcal{C}}}$ which appears as a summand of $T_{s_j} \cdot C$. Note that $f_{i,j}^{\mathrm{KL}_{\mathcal{C}}}(C_w) = C_{f_{i,j}^{S_n}(w)}$.*

Now, we immediately have a definition of generalized τ-invariant for each basis element in $\mathrm{KL}_{\mathcal{C}}$ simply by substituting $f_{i,j}^{\mathrm{KL}_{\mathcal{C}}}$ in Definition 1. In fact we can compare generalized τ-invariant of basis elements and permutations or tableaux by using the appropriate $f_{i,j}$ maps on each side of the equations in Definition 1.

Theorem 7 (Robinson–Schensted for left cells) *Let C_w be a basis element in $\mathrm{KL}_{\mathcal{C}}$. Then, $P(w)$, the left Robinson–Schensted tableau for w, is the unique standard tableau on n boxes such that $\tau_g(P(W)) = \tau_g(C_w)$.*

Thus, suppose that $\mathbf{p}$ is some partition of n. (This choice equivalent to the choice of a Young diagram with n boxes.) Let $\mathrm{KL}_{\mathbf{p}}$ be the left cell for this shape with permutation and tableau labeling stripped away. (Recall that $\mathrm{KL}_{\mathbf{p}}$ is uniquely determined as a based representation due to Theorem 6 regardless of the particular left cell from which it originally derived.) Then, given some left cell basis element $C \in \mathrm{KL}_{\mathbf{p}}$, there is a unique standard tableau Y_C such that $\tau_g(C) = \tau_g(Y_C)$. Furthermore, this is the same tableau that we would have obtained by taking the left tableau for the permutation label of C before it was removed. The parameterization of left cell basis elements by standard tableaux on the shape $\mathbf{p}$ is in this sense canonical.

6 The Symmetric Group Action on sl_3-Webs

The sl_3 spider, introduced by Kuperberg (1996) and subsequently studied by many others (Khovanov and Kuperberg (1999); Kim (2003); Morrison (2007); Murakami et al. (1998)) is a diagrammatic, braided monoidal category encoding the representation theory of $U_q(sl_3)$. The objects in this category are tensor products of V^+ and V^-, the three-dimensional representations of $\mathcal{U}_q(\mathfrak{sl}_3)$, but these are encoded as finite strings in the alphabet $\{+, -\}$, including the empty string. The morphisms are intertwining maps, which are represented by $\mathbb{Z}[q, q^{-1}]$-linear combinations of certain graphs called webs which we will describe in a moment. (See Figure 1 for an example of a web.) Webs are oriented trivalent graphs drawn in a

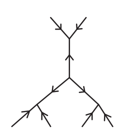
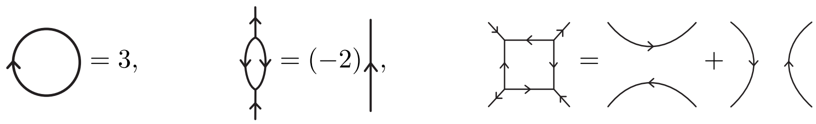


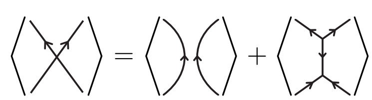
Fig. 1: A web in $\mathrm{Hom}(++++\,,\,--)$

square region with boundary points lying on the top and bottom of that region. Edges incident on the boundary points have orientations compatible with the source and target words; edges pointing upward and downward are labeled by $+$ and $-$ respectively. We read webs from bottom to top. All trivalent vertices are either sources or sinks. Webs are also subject to the relations in Equation (6) below, which are often referred to as the circle, bigon, and square relations. (Reduced webs are those with no circles, squares or bigons. For simplicity, we've given these relations in the classical limit.)



(6)

As mentioned in the introduction, there is a Hecke algebra action on invariant webs for tensor powers of V^+. We will work in the classical limit, where this becomes a symmetric group action. In this case, a crossing in the symmetric group reduces to the morphism given in Equation (7). In other words, to act on some invariant web by s_i, we attach the diagram on the right to the i and $i+1$ vertices of the web. This gives a sum of two webs which may or may not be reduced. Reducing summands may give a sum of more than two reduced webs. Petersen et al. (2009) prove that this action is, up to isomorphism, the irreducible S_{3n}-representation corresponding to the partition $[n, n, n]$. (See Theorem 4 above.) For an example of computing the action of a permutation on a web, see Figure 7 on Page 871. Because this computation involves two crossings, each of which can be "smoothed" in two ways as per Figure 2, we initially obtain four web terms. Subsequently, we reduce out squares and bigons and simplify.



(7)

Fig. 2: The symmetric group crossing morphism in the classical sl_3-spider

From now on, we will sometimes omit orientations in our graphs. Orientations are uniquely determined by the fact that edges point away from the bottom boundary.

7 Kuperberg's Bijection

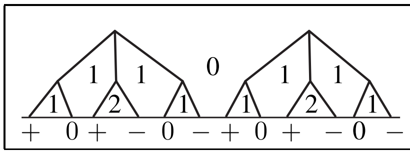
Kuperberg introduced a bijection between webs and dominant lattice paths in the weight lattice of $\mathfrak{sl}_3$. For webs with $3n$ source vertices, this may be interpreted as a bijection between standard tableaux of shape $[n, n, n]$ and reduced webs (Petersen et al. (2009)).

The map sends each web to a *Yamanouchi word* which is then used to build a standard tableau. Given a tableau T of shape $[n, n, n]$, the Yamanouchi word $y_T = y_1 y_2 \cdots y_{3n-1} y_{3n}$ is a string of symbols in the alphabet $\{+, 0, -\}$ where

$$y_i = \begin{cases} + & \text{if } i \text{ is in the top row of } T, \\ 0 & \text{if } i \text{ is in the middle row of } T, \text{ and} \\ - & \text{if } i \text{ is in the bottom row of } T. \end{cases}$$

As an example, the Yamanouchi word for the tableau T in Figure 4 is $y_T = +0+-0-$. Yamanouchi words corresponding to standard fillings of shape $[n, n, n]$ are completely characterized by two properties: They have n of each symbol, and at any point in the word, the number of $+$'s is greater than or equal to the number of 0's which is greater than or equal to the number of $-$'s. These words are called balanced. The algorithm to build Yamanouchi words from webs is as follows:

Start with a reduced web W on $3n$ source vertices aligned along a horizontal line with W drawn in the upper half plane. The horizontal line containing the vertices and edges in the web divides the upper half plane into faces, with one infinite face; label the infinite face 0. Label each additional face with the minimum number of edges that a path must cross to reach this face from the infinite face. Under each base vertex, write $+$, 0 or $-$ to indicate that the labels on the faces directly above the vertex increase, stay the same or decrease as we read from left to right. The string under the horizontal line is the Yamanouchi word of a standard $[n, n, n]$ tableau T. Complete the algorithm by writing down T. We demonstrate this computation in Figure 3.



1	3	7	9
2	5	8	11
4	6	10	12

Fig. 3: A web and its Yamanouchi word obtained from depth labels, with the corresponding standard tableau.

Khovanov and Kuperberg (1999) introduced a method for computing the inverse map, but Tymoczko (2012) recently developed a much simpler approach in terms of m-*diagrams*. Given a tableau T, construct an m-diagram m_T as follows:

Draw a horizontal line with $3n$ equally spaced dots labeled from left to right with the numbers $1, \ldots, 3n$. This line forms the lower boundary for the diagram, and all arcs will lie above it. Starting with the smallest number j on the second row, draw a semi-circular arc connecting j to its nearest unoccupied neighbor i to the left that appears in the first row. The arcs (i, j) are the left arcs in the m-diagram. Starting with the smallest number k on the bottom row, draw a semi-circular arc connecting k to its nearest neighbor j to the left that appears in the second row and does not already have an arc coming to it from the left. The arcs (j, k) are the right arcs of the m-diagram. The collection of left arcs is nonintersecting as is the collection of right arcs, but left arcs can intersect right arcs. Figure 4 shows an example of an m-diagram.

1	3
2	5
4	6

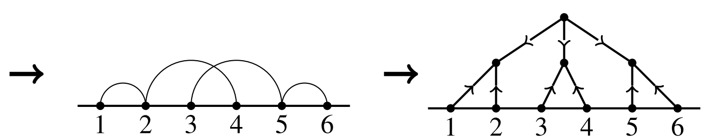


Fig. 4: The m-diagram and web corresponding to a tableau.

From an m-diagram m_T for T, the following straightforward process transforms m_T into an irreducible web W_T. Figure 4 shows a web corresponding to an m-diagram.

At each boundary vertex where two semi-circular arcs meet, replace the portion of the diagram in a small neighborhood of the vertex with a 'Y' shape as shown in Figure 5. Orient all arcs away from the boundary so that the branching point of each 'Y' becomes a source. Finally replace any 4-valent intersection point of a left arc and a right arc with a pair of trivalent vertices as shown in Figure 6. There is a unique way to do this preserving orientation of incoming arcs.

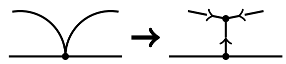


Fig. 5: Modifying the middle vertex of an m.

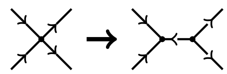


Fig. 6: Replacing a 4-valent vertex with trivalent vertices.

8 Kuperberg's Bijection as Robinson–Schensted Analogue

Recall that $\tau(T)$ is the set of all simple transpositions s_i for which i is in a row above $i+1$ in the tableau T. Our first observation is that $\tau(T)$ is completely determined by looking at W_T.

Lemma 7 *Given a standard tableau T and its associated web W_T, the set $\tau(T)$ consists of all transpositions s_i for which boundary vertices i and $i+1$ are directly connected to the same internal vertex in W_T. Furthermore, $s_i \in \tau(T)$ if and only if $s_i \cdot W_T = -W_T$.*

Thus, we can define $\tau(W)$ for any web W, and this notion agrees with our existing definitions for tableaux and Kazhdan–Lusztig left cell basis elements. In addition, we can define the set $D_{i,j}^{\mathrm{web}}$ as reduced webs whose τ-invariants contain s_i but not s_j.

Lemma 8 *Let $T \in D_{i,j}^{\mathrm{YT}}$. Then $s_j \cdot W_T = W_T + W_{f_{i,j}^{\mathrm{YT}}(T)} + O$ where O is a $\mathbb{Z}$-linear combination of reduced webs, none of which lies in $D_{j,i}^{\mathrm{web}}$.*

Using Lemma 8, we get a definition for $f_{i,j}^{\mathrm{web}}$, which is essentially the same as Definition 8 from the Kazhdan–Lusztig left cell setting:

Definition 9 *Given a reduced web $W \in D_{i,j}^{\mathrm{web}}$, let $f_{i,j}^{\mathrm{web}}(W)$ be the unique reduced web in $D_{j,i}^{\mathrm{web}}$ which appears as a summand in $s_j \cdot W$. This defines a bijection $f_{i,j}^{\mathrm{web}} : D_{i,j}^{\mathrm{web}} \to D_{j,i}^{\mathrm{web}}$ with inverse $f_{j,i}^{\mathrm{web}} : D_{j,i}^{\mathrm{web}} \to D_{i,j}^{\mathrm{web}}$.*

Of course, $f_{i,j}^{\mathrm{web}}(W_T) = W_{f_{i,j}^{\mathrm{YT}}(T)}$. We then obtain generalized τ-invariants for reduced webs by replacing the $f_{i,j}^{\mathrm{YT}}$ maps in Definition 1 with $f_{i,j}^{\mathrm{web}}$. Notice that the definition of the generalized τ-invariant for a reduced web is essentially identical to the *in situ* version for Kazhdan–Lusztig left cell basis elements in terms of the S_n-action. Combining Lemmas 7 and 8 with Theorem 1 brings us to the main theorem of the paper:

Theorem 8 (Robinson–Schensted for Webs) *Kuperberg's bijection carries a reduced web W on $3n$ source vertices to the unique $[n,n,n]$ standard tableau T satisfying $\tau_g(T) = \tau_g(W)$.*

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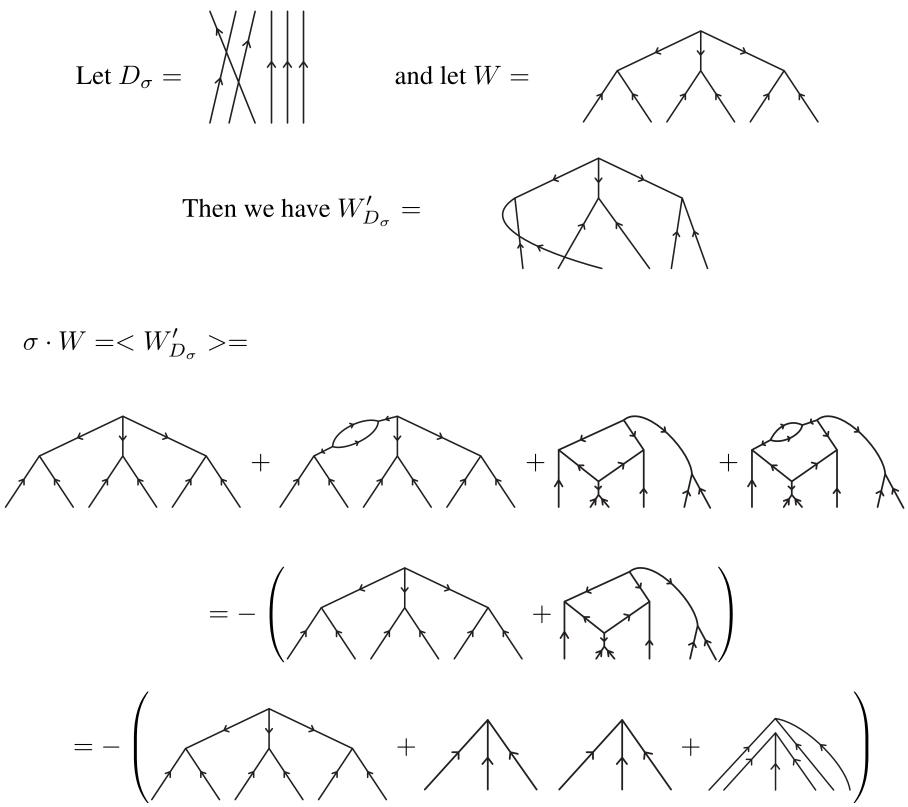

Fig. 7: Computing the action of a permutation on a web.

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