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Divisors on graphs, Connected flags, and Syzygies

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Abstract. We study the binomial and monomial ideals arising from linear equivalence of divisors on graphs from the point of view of Gröbner theory. We give an explicit description of a minimal Gröbner basis for each higher syzygy module. In each case the given minimal Gröbner basis is also a minimal generating set. The Betti numbers of I_G and its initial ideal (with respect to a natural term order) coincide and they correspond to the number of "connected flags" in G. Moreover, the Betti numbers are independent of the characteristic of the base field.

Résumé. Nous étudions les idéaux monômiaux et binomiaux résultant de l'équivalence linéaire de diviseurs sur les graphes du point de vue de la théorie de Gröbner. Nous donnons une description explicite d'une base de Gröbner minimale pour chaque module engendré par une syzygie d'ordre supérieur. Dans chaque cas, cette base de Gröbner minimale est aussi une ensemble generateur minimal. Les nombres de Betti de I_G et son idéal initial coïncident et correspondent au nombre de $\ll$ drapeaux connexes $\gg$ de G. En particulier, les nombres de Betti sont indépendants de la caractéristique du corps de référence.

Keywords: Graph, divisors, chip-firing, Gröbner bases, Betti numbers, connected flags.

1 Introduction

The theory of divisors on finite graphs can be viewed as a discrete version of the analogous theory on Riemann surfaces. This notion arises in different fields of research including the study of "abelian sandpiles" ([Dha90, Gab93]), the study of component groups of Néron models of Jacobians of algebraic curves ([Ray70, Lor89]), and the theory of chip-firing games on graphs ([Big97]). Riemann-Roch theory for finite graphs (and generalizations to tropical curves) is developed in this setting ([BN07, GK08, MZ08]).

We are interested in the linear equivalence of divisors on graphs from the point of view of commutative algebra. Associated to every graph G there is a canonical binomial ideal I_G which encodes the linear equivalences of divisors on G. Let R denote the polynomial ring with one variable associated to each vertex. For any two effective divisors $D_1 \sim D_2$ there is a binomial $\mathbf{x}^{D_1} - \mathbf{x}^{D_2}$. The ideal $I_G \subset R$ is generated by all such binomials. Two effective divisors are linearly equivalent if and only if their associated monomials are equal in R/I_G. This ideal is already implicitly defined in Dhar's seminal statistical physics paper [Dha90] ; R/I_G is the "operator algebra" defined there. To our knowledge, this ideal (more precisely, an

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affine piece of it) was first introduced in [CRS02] to address computational questions in chip-firing dynamics using Gröbner basis. From a purely computational point of view there are now much more efficient methods available (see, e.g., [BS13] and references therein). However this ideal seems to encode a lot of interesting information about G and its linear systems. Some of the algebraic properties of I_G (and its generalization for directed graphs) are studied in [PPW11]. In [MS13a], Manjunath and Sturmfels relate Riemann-Roch theory for finite graphs to Alexander duality in commutative algebra using this ideal.

In this paper we study the syzygies and free resolutions of the ideals I_G and $\mathrm{in}(I_G)$ from the point of view of Gröbner theory. Here $\mathrm{in}(I_G)$ denotes the initial ideal with respect to a natural term order which is defined after distinguishing a vertex q (see Definition 2.1). When G is a complete graph, the syzygies and Betti numbers of the ideal $\mathrm{in}(I_G)$ are studied by Postnikov and Shapiro in [PS04]. Again for complete graphs, Manjunath and Sturmfels in [MS13a] study the ideal I_G and show that the Betti numbers coincide with the Betti numbers of $\mathrm{in}(I_G)$. Finding minimal free resolutions for a general graph G was stated as an open problem in both [PS04] and [MS13a] (also in [PPW11], where a conjecture is formulated). It was not even known whether the Betti numbers for a general graph depend on the characteristic of the base field or not.

We construct free resolutions for both $\mathrm{in}(I_G)$ and I_G for a general graph G. Indeed we describe, combinatorially, the minimal Gröbner bases for all higher syzygy modules of I_G and $\mathrm{in}(I_G)$. In each case the minimal Gröbner basis is also a minimal generating set and the given resolution is minimal. In particular the Betti numbers of $\mathrm{in}(I_G)$ and I_G coincide. This gives a positive answer to [CHT06, Question 1.1] for ideal I_G. For a complete graph the minimal free resolution for $\mathrm{in}(I_G)$ is nicely structured by a Scarf complex. The resolution for I_G when G a tree is given by a Koszul complex since I_G is a complete intersection. A more conceptual and geometric proof for a general graph G will be given in [MS13b].

The description of the generating sets and the Betti numbers is in terms of the "connected flags" of G. Fix a vertex $q \in V(G)$ and an integer k. A *connected k-flag* of G (based at q) is a strictly increasing sequence $U_1 \subsetneq U_2 \subsetneq \cdots \subsetneq U_k = V(G)$ such that $q \in U_1$ and all induced subgraphs on vertex sets U_i and $U_{i+1} \backslash U_i$ are connected. Associated to any connected k-flag one can assign a "partial orientation" on G (Definition 3.3). Two connected k-flags are considered equivalent if the associated partially oriented graphs coincide. The Betti numbers correspond to the numbers of the connected flags up to this equivalence. We give a bijective map between the connected flags of G and the minimal Gröbner bases for higher syzygy modules of I_G and $\mathrm{in}(I_G)$. For a complete graph all flags are connected and all distinct flags are inequivalent. So in this case the Betti numbers are simply the face numbers of the order complex of the poset of those subsets of $V(G)$ that contain q (ordered by inclusion). These numbers can be described using classical Stirling numbers (see Example 4.6). Hence our results directly generalize the analogous results in [PS04] and [MS13a]. Analogous results with different methods were obtained independently in [MSW12] and in [DS12].

The paper is structured as follows. In § 2 we fix our notation and provide the necessary background from the theory of divisors on graphs. We also define the ideal I_G and the natural $\mathrm{Pic}(G)$-grading and a term order $<$ on the polynomial ring relevant to our setting. In § 2.2 we quickly recall some basic notions from commutative algebra. Our main goal is to fix our notation for Schreyer's algorithm for computing higher syzygies, which is slightly different from what appears in the existing literature but is more convenient for our application. In § 3 we define connected flags and their equivalence relation. In § 4 we study the free resolution and higher syzygies of our ideals from the point of view of Gröbner theory, and as a corollary we give our description of the graded Betti numbers. In § 5 we describe some connections with the theory of reduced divisors. We refer to [MS12] for proofs and more details.

2 Definitions and background

2.1 Graphs and divisors

Throughout this paper, a *graph* means a finite, connected, unweighted multigraph with no loops. As usual, the set of vertices and edges of a graph G are denoted by $V(G)$ and $E(G)$. We set $n = |V(G)|$. A *pointed graph* (G, q) is a graph together with a choice of a distinguished vertex $q \in V(G)$.

Let $\mathrm{Div}(G)$ be the free abelian group generated by $V(G)$. An element of $\mathrm{Div}(G)$ is written as $\sum_{v \in V(G)} a_v(v)$ and is called a *divisor* on G. The coefficient a_v in D is also denoted by $D(v)$. A divisor D is called *effective* if $D(v) \geq 0$ for all $v \in V(G)$. The set of effective divisors is denoted by $\mathrm{Div}_+(G)$. We write $D \leq E$ if $E - D \in \mathrm{Div}_+(G)$. For $D \in \mathrm{Div}(G)$, let $\deg(D) = \sum_{v \in V(G)} D(v)$. For $D_1, D_2 \in \mathrm{Div}(G)$, the divisor $E = \max(D_1, D_2)$ is defined by $E(v) = \max(D_1(v), D_2(v))$ for $v \in V(G)$.

We denote by $\mathcal{M}(G)$ the group of integer-valued functions on the vertices. For $A \subseteq V(G)$, $\chi_A \in \mathcal{M}(G)$ denotes the $\{0,1\}$-valued characteristic function of A. The *Laplacian operator* $\Delta : \mathcal{M}(G) \to \mathrm{Div}(G)$ is defined by

$$\Delta(f) = \sum_{v \in V(G)} \sum_{\{v,w\} \in E(G)} (f(v) - f(w))(v) \ .$$

The group of *principal divisors* is defined as the image of the Laplacian operator and is denoted by $\mathrm{Prin}(G)$. It is easy to check that $\mathrm{Prin}(G) \subseteq \mathrm{Div}^0(G)$ where $\mathrm{Div}^0(G)$ denotes the set consisting of divisors of degree zero. The quotient $\mathrm{Pic}^0(G) = \mathrm{Div}^0(G)/\mathrm{Prin}(G)$ is a finite group whose cardinality is the number of spanning trees of G (see, e.g., [BS13] and references therein). The full *Picard group* of G is defined as

$$\mathrm{Pic}(G) = \mathrm{Div}(G)/\mathrm{Prin}(G)$$

which is isomorphic to $\mathbb{Z} \oplus \mathrm{Pic}^0(G)$. Since principal divisors have degree zero, the map $\deg : \mathrm{Div}(G) \to \mathbb{Z}$ descends to a well-defined map $\deg : \mathrm{Pic}(G) \to \mathbb{Z}$. Two divisors D_1 and D_2 are called *linearly equivalent* if they become equal in $\mathrm{Pic}(G)$. In this case we write $D_1 \sim D_2$. The *linear system* $|D|$ of D is defined as the set of effective divisors that are linearly equivalent to D.

To an ordered pair of *disjoint* subsets $A, B \subseteq V(G)$ we assign an effective divisor

$$D(A, B) = \sum_{v \in A} |\{w \in B : \{v, w\} \in E(G)\}|(v) \ .$$

In other words, the support of $D(A, B)$ is a subset of A and for $v \in A$ the coefficient of (v) in $D(A, B)$ is the number of edges between v and B.

Let K be a field and let $R = K[\mathbf{x}]$ be the polynomial ring in the n variables $\{x_v : v \in V(G)\}$. Any effective divisor D gives rise to a monomial

$$\mathbf{x}^D := \prod_{v \in V(G)} x_v^{D(v)} \ .$$

Associated to every graph G there is a canonical ideal which encodes the linear equivalences of divisors on G. Our main object study is the ideal

$$I_G := \langle \mathbf{x}^{D_1} - \mathbf{x}^{D_2} : D_1 \sim D_2 \text{ both effective divisors} \rangle$$

which was introduced in [CRS02].

Once we fix a vertex q, there is a natural monomial order that gives rise to a particularly nice Gröbner basis for I_G. This term order was first introduced in [CRS02]. Fix a pointed graph (G, q). Consider a total ordering of the set of variables $\{x_v : v \in V(G)\}$ compatible with the distances of vertices from q in G :

$$\operatorname{dist}(w, q) < \operatorname{dist}(v, q) \implies x_w < x_v \, . \tag{1}$$

Here, the distance between two vertices in a graph is the number of edges in a shortest path connecting them. The above ordering can be thought of an ordering on vertices induced by running the breadth-first search algorithm starting at the root vertex q.

Definition 2.1 *We denote by $<$ the degree reverse lexicographic ordering on $R = K[\mathbf{x}]$ induced by the total ordering on the variables given in* (1)*.*

Throughout this paper $\operatorname{in}(I_G)$ denotes the initial ideal of I_G with respect to this term order. Note that $\operatorname{in}(I_G)$ is denoted by M_G in [PS04].

2.2 Syzygies and Betti numbers

In this subsection we quickly recall some basic notions from commutative algebra in order to fix our notation. We refer to standard books (e.g. [Eis95, GP08]) for more details.

Let K be any field and let $R = K[\mathbf{x}]$ be the polynomial ring in n variables graded by an abelian group A. The degree map will be denoted by $\deg$. Let M be a graded submodule of a free module and fix a *module ordering* $<_0$ extending the monomial ordering $<$ on R. Assume that the finite totally ordered set $(\mathbf{G}, \prec)$ forms a Gröbner basis for $(M, <_0)$ consisting of homogeneous elements. Let F_0 be the free module generated by $\mathbf{G}$. For $g \in \mathbf{G}$ we let the formal symbol $[g]$ denote the corresponding generator for F_0 ; each element of F_0 can be written as a sum of these formal symbols with coefficients in R. There is a natural surjective homomorphism

$$\varphi_0 : F_0 \longrightarrow M$$

sending $[g]$ to g for each $g \in \mathbf{G}$. Moreover, we enforce this homomorphism to be graded (or homogeneous of degree 0) by defining $\deg([g]) := \deg(g)$ for all $g \in \mathbf{G}$.

By definition the syzygy module of M with respect to $\mathbf{G}$, denoted by $\operatorname{syz}(\mathbf{G})$, is the kernel of this map. Let $\operatorname{syz}_0(\mathbf{G}) := M$ and $\operatorname{syz}_1(\mathbf{G}) := \operatorname{syz}(\mathbf{G})$. For $i > 1$ the higher syzygy modules are defined as $\operatorname{syz}_i(\mathbf{G}) := \operatorname{syz}(\operatorname{syz}_{i-1}(\mathbf{G}))$.

We now discuss a method to compute a Gröbner basis for $\operatorname{syz}(\mathbf{G})$. One can "pull back" the module ordering $<_0$ along φ_0 to get a compatible module ordering $<_1$ on F_0 ; for $f, h \in \mathbf{G}$ define

$$\mathbf{x}^{\beta}[h] <_1 \mathbf{x}^{\alpha}[f] \Leftrightarrow \begin{cases} \operatorname{LM}(\mathbf{x}^{\beta} h) <_0 \operatorname{LM}(\mathbf{x}^{\alpha} f) \\ \text{or} \\ \operatorname{LM}(\mathbf{x}^{\beta} h) = \operatorname{LM}(\mathbf{x}^{\alpha} f) \quad \wedge \quad f \prec h. \end{cases} \tag{2}$$

To simplify the notation we assume the leading coefficients of all elements of $\mathbf{G}$ are 1. For a pair of elements $f \prec h$ of $\mathbf{G}$ assume

$$\mathrm{LM}(f) = \mathbf{x}^{\alpha(f)}[e] \quad \text{and} \quad \mathrm{LM}(h) = \mathbf{x}^{\alpha(h)}[e]$$

for some $e \in E$. Since $\mathbf{G}$ is a Gröbner basis, setting $\gamma(f,h) := \max(\alpha(f), \alpha(h))$, we have the "standard representation" :

$$\mathrm{spoly}(f,h) = \mathbf{x}^{\gamma(f,h)-\alpha(f)} f - \mathbf{x}^{\gamma(f,h)-\alpha(h)} h = \sum_{g \in \mathbf{G}} a_g^{(f,h)} g \tag{3}$$

for some polynomials $a_g^{(f,h)} \in R$. We set

$$s(f,h) = \mathbf{x}^{\gamma(f,h)-\alpha(f)}[f] - \mathbf{x}^{\gamma(f,h)-\alpha(h)}[h] - \sum_{g \in \mathbf{G}} a_g^{(f,h)}[g] \in F_0 \ . \tag{4}$$

Theorem 2.2 (Schreyer [Sch80], [Eis95]) *The set*

$$\mathcal{S}(\mathbf{G}) = \{ s(f,h) : \ f,h \in \mathbf{G}\, ,\ f \prec h\, ,\ \mathrm{LM}(f) = \mathbf{x}^{\alpha(f)}[e]\, ,\ \mathrm{LM}(h) = \mathbf{x}^{\alpha(h)}[e] \textit{ for some } e \in E \}$$

forms a homogeneous Gröbner basis for $(\mathrm{syz}(\mathbf{G}), <_1)$.

To read the Betti numbers for M one needs to find a *minimal generating set* for the syzygy modules. In general the set $\mathcal{S}(\mathbf{G})$ is far from being even a *minimal Gröbner basis*. However there exist some criterions to find a subset $\mathcal{S}_{\min}(\mathbf{G})$ of $\mathcal{S}(\mathbf{G})$ which forms a minimal Gröbner basis for $(\mathrm{syz}(\mathbf{G}), <_1)$; see, e.g., [MS12, Lemma 3.4]. Moreover, Theorem 2.2 gives rise to Algorithm 1 for computing free resolutions. The following result gives a general sufficient criterion for an ideal to have the same Betti numbers as its initial ideal.

Theorem 2.3 *If the constructed resolution by Schreyer's algorithm is a* minimal *graded free resolution then* $\beta_{i,\mathsf{j}}(M) = \beta_{i,\mathsf{j}}(\mathrm{in}(M))$ *for all* $i \geq 0$ *and* $\mathsf{j} \in \mathsf{A}$.

3 Connected flags on graphs

3.1 Connected flags, partial orientations, and divisors

From now on we fix a pointed graph (G,q) and we let $n = |V(G)|$. Consider the poset

$$\mathfrak{C}(G,q) := \{ U \subseteq V(G) : \ q \in U \}$$

ordered by inclusion. The following special chains of this poset arise naturally in our setting.

Definition 3.1 *Fix an integer* $1 \leq k \leq n$. *A* connected k-flag *of* (G,q) *is a (strictly increasing) sequence* $\mathcal{U}$ *of subsets of* $V(G)$

$$U_1 \subsetneq U_2 \subsetneq \cdots \subsetneq U_k = V(G)$$

such that $q \in U_1$ *and, for all* $1 \leq i \leq k-1$, *both* $G[U_i]$ *and* $G[U_{i+1} \backslash U_i]$ *are connected.*

The set of all connected k-flags of (G,q) will be denoted by $\mathfrak{F}_k(G,q)$.

Input:
Graded polynomial ring $R = K[\mathbf{x}]$,
Monomial ordering $<$ on R ,
Graded submodule M of the free R-module F_{-1} generated by formal symbols $\{[e]\}_{e\in E}$,
Module ordering $<_0$ on F_{-1} extending the monomial ordering $<$,
Finite set $\mathbf{G}$ forming a homogeneous Gröbner basis for $(M, <_0)$.
Output:
A free resolution : $0 \to \cdots \to F_i \xrightarrow{\varphi_i} F_{i-1} \to \cdots \to F_0 \xrightarrow{\varphi_0} M \to 0$.
Initialization :
$\mathbf{G}_0 := \mathbf{G}$;
$F_0 :=$ free R-module generated by formal symbols $\{[g]\}_{g\in\mathbf{G}_0}$; **Output** F_0 ;
$\varphi_0 : F_0 \to M \subseteq F_{-1}$ defined by $[g] \mapsto g$ for each $g \in \mathbf{G}_0$; **Output** φ_0 ;
$i = 0$;
while $F_i \neq 0$ **do**
  $\prec_i$: arbitrary total ordering on $\mathbf{G}_i$;
  $<_{i+1}$: module ordering on F_i obtained from $<_i$ on F_{i-1} (as in (2)) ;
  $\mathbf{G}_{i+1} := \mathcal{S}_{\min}(\mathbf{G}_i) \subset F_i$, a minimal Gröbner basis of $(\mathrm{syz}_{i+1}(\mathbf{G}), <_{i+1})$ (as in Theorem 2.2) ;
  F_{i+1} := free R-module generated by formal symbols $\{[u]\}_{u\in\mathbf{G}_{i+1}}$; **Output** F_{i+1} ;
  $\varphi_{i+1} : F_{i+1} \to F_i$ defined by $[u] \mapsto u$ for each $u \in \mathbf{G}_{i+1}$; **Output** φ_{i+1} ; $i \leftarrow i + 1$;
end

Algorithm 1: Algorithm for computing a free resolution of M (Schreyer's algorithm)

Remark 3.2 For a complete graph, $\mathfrak{F}_k(G, q)$ is simply the order complex of $\mathfrak{C}(G, q)$, but in general $\mathfrak{F}_k(G, q)$ is not a simplicial complex.

Definition 3.3 *Given $\mathcal{U} \in \mathfrak{F}_k(G, q)$ we define :*
(a) a "partial orientation" of G by orienting edges from U_i to $U_{i+1}\backslash U_i$ *(for all $1 \le i \le k-1$) and leaving all other edges unoriented. We denote the resulting partially oriented graph by $G(\mathcal{U})$.*
(b) an effective divisor $D(\mathcal{U}) \in \mathrm{Div}(G)$ given by $D(\mathcal{U}) := \sum_{i=1}^{k-1} D(U_{i+1}\backslash U_i, U_i)$.

Remark 3.4 It is easy to check that $D(\mathcal{U}) = \sum_{v\in V(G)} (\mathrm{indeg}_{G(\mathcal{U})}(v))(v)$, where $\mathrm{indeg}_{G(\mathcal{U})}(v)$ denotes the number of oriented edges directed *to* v in $G(\mathcal{U})$.

3.2 Total ordering on $\mathfrak{F}_k(G, q)$

We endow each $\mathfrak{F}_k(G, q)$ with a *total orderings* $\prec_k$ for all $1 \le k \le n$. These total orderings are compatible with each other for different values of $1 \le k \le n$.

Let $\preceq$ denote the ordering on $\mathfrak{C}^{\mathrm{op}}(G, q)$ given by reverse inclusion :

$$U \preceq V \iff U \supseteq V \ .$$

Definition 3.5 *We fix, once and for all, a* total ordering *extending $\preceq$. By a slight abuse of notation, $\preceq$ will be used to denote this total ordering extension. In particular $\prec$ will denote the associated strict total order.*

We consider one of the natural “lexicographic extensions” of $\prec$ to the set of connected k-flags.

Definition 3.6 *For $\mathcal{U} \neq \mathcal{V}$ in $\mathfrak{F}_k(G, q)$ written as*

$$\mathcal{U}: \; U_1 \subsetneq U_2 \subsetneq \cdots \subsetneq U_k = V(G)$$

$$\mathcal{V}: \; V_1 \subsetneq V_2 \subsetneq \cdots \subsetneq V_k = V(G)$$

we say $\mathcal{U} \prec_k \mathcal{V}$ if for the maximum $1 \leq \ell \leq k-1$ with $U_\ell \neq V_\ell$ we have $\mathcal{U}_\ell \prec \mathcal{V}_\ell$.
As usual, we write $\mathcal{U} \preceq_k \mathcal{V}$ if and only if $\mathcal{U} \prec_k \mathcal{V}$ or $\mathcal{U} = \mathcal{V}$.

Lemma 3.7 *$(\mathfrak{F}_k(G, q), \preceq_k)$ is a totally ordered set.*

It is easy to find two different connected k-flags having identical associated partially oriented graphs. This motivates the following definition.

Definition 3.8 *Two k-flags $\mathcal{U}, \mathcal{V} \in \mathfrak{F}_k(G, q)$ are called equivalent if the associated partially oriented graphs $G(\mathcal{U})$ and $G(\mathcal{V})$ coincide.*

Notation 1 *The set of all equivalence classes in $\mathfrak{F}_k(G, q)$ will be denoted by $\mathfrak{E}_k(G, q)$. The set $\mathfrak{S}_k(G, q)$ denotes the collection of minimal representatives of the classes in $\mathfrak{E}_k(G, q)$ with respect to $\prec_k$.*

Given an element in $\mathfrak{S}_k(G, q)$ there is a canonical way to obtain two related elements in $\mathfrak{S}_{k-1}(G, q)$.

Definition 3.9 *Given $\mathcal{U} \in \mathfrak{F}_k(G, q)$, the elements $\mathcal{U}^{(1)}, \mathcal{U}^{(2)} \in \mathfrak{F}_{k-1}(G, q)$ are obtained from $\mathcal{U}$ by removing the 1^{st} and 2^{nd} elements in the following appropriate sense. Let*

$$\mathcal{U}: \; U_1 \subsetneq U_2 \subsetneq \cdots \subsetneq V(G)\ .$$

(a) $\mathcal{U}^{(1)}$ will denote

$$U_2 \subsetneq U_3 \subsetneq U_4 \subsetneq \cdots \subsetneq V(G)\ .$$

(b) $\mathcal{U}^{(2)}$ will denote

$$\begin{cases} U_1 \subsetneq U_3 \subsetneq U_4 \subsetneq \cdots \subsetneq V(G), & \text{if } G[U_3 \backslash U_1] \text{ is connected}\,; \\ \text{or} & \\ (U_1 \cup (U_3 \backslash U_2)) \subsetneq U_3 \subsetneq U_4 \subsetneq \cdots \subsetneq V(G), & \text{if } G[U_3 \backslash U_1] \text{ is not connected.} \end{cases}$$

Remark 3.10 [MS12, Section 6.1] Assume that $\mathcal{U} \in \mathfrak{S}_k(G, q)$. Let $G_{/\mathcal{U}}$ be the graph obtained from G by contracting the unoriented edges of $G(\mathcal{U})$ and let $\phi : G \to G_{/\mathcal{U}}$ be the contraction map. More precisely, $G_{/\mathcal{U}}$ is the graph on the vertices $u_1, \ldots, u_k$ corresponding to the collection $(U_i \backslash U_{i-1})_{i=1}^k$, i.e. $u_i = \phi(U_i \backslash U_{i-1})$. For any edge between $U_i \backslash U_{i-1}$ and $U_j \backslash U_{j-1}$ there is an edge between u_i and u_j. The contraction map $\phi : G \to G_{/\mathcal{U}}$ induces the maps

(i) $\phi_* : \mathrm{Div}(G) \to \mathrm{Div}(G_{/\mathcal{U}})$ with $\phi_*(\sum_{v \in V(G)} a_v(v)) = \sum_{v \in V(G)} a_v(\phi(v))$. In particular, a total ordering $u_1, \ldots, u_k$ of $V(G_{/\mathcal{U}})$ gives a total ordering on the collection of subsets $(U_i \backslash U_{i-1})_{i=1}^k$ of $V(G)$. By Definition 3.3 we get a divisor D' on $G_{/\mathcal{U}}$ and a divisor D on G, and $\phi_*(D) = D'$. In other words, such a divisor D' has a canonical section.

(ii) $\phi^* : \mathfrak{S}_s(G_{/\mathcal{U}}, u_1) \to \mathfrak{S}_s(G, q)$ with $\phi^*(\mathcal{V}') = \mathcal{V}$ where $V_j = \bigcup_{u_i \in V'_j} (U_i \backslash U_{i-1})$.

4 Minimal free resolution and Betti numbers for I_G and $\mathrm{in}(I_G)$

Let K be a field and let $R = K[\mathbf{x}]$ be the polynomial ring in n variables $\{x_v : v \in V(G)\}$. Recall that $K[\mathbf{x}]$ has a natural A-grading, where $\mathsf{A} = \mathbb{Z}$ or $A = \mathrm{Pic}(G)$ and I_G is also A-graded. Let the monomial ordering $<$ on R be as in Definition 2.1.

The following theorem gives a generalization of [CRS02, Theorem 14]. Indeed [CRS02, Theorem 14] can be rephrased as providing a bijection between $\mathfrak{S}_2(G,q)$ and $\mathbf{G}(G,q)$.

Theorem 4.1 *Fix a pointed graph (G,q) and let $\mathsf{A} = \mathbb{Z}$ or $\mathsf{A} = \mathrm{Pic}(G)$. For each $k \geq 0$ there exists a natural injection*

$$\psi_k : \mathfrak{S}_{k+2}(G,q) \hookrightarrow \mathrm{syz}_k(\mathbf{G}(G,q))$$

such that

(i) For some module ordering $<_k$, the set $\mathbf{G}_k(G,q) := \mathrm{Image}(\psi_k)$ forms a minimal A-homogeneous Gröbner basis of $(\mathrm{syz}_k(\mathbf{G}(G,q)), <_k)$,

(ii) For $\mathcal{U} \in \mathfrak{S}_{k+2}(G,q)$ of the form $U_1 \subsetneq U_2 \subsetneq \cdots \subsetneq V(G)$ we have

$$\mathrm{LM}(\psi_k(\mathcal{U})) = \mathbf{x}^{D(U_2 \setminus U_1, U_1)}[\psi_{k-1}(\mathcal{U}^{(1)})] \ , \tag{5}$$

(iii) The set $\psi_k(\mathfrak{S}_{k+2}(G,q))$ minimally generates $\mathrm{syz}_k(\mathbf{G}(G,q))$.

Sketch of proof : Here we list the key steps of the proof. For a complete proof we refer to [MS12]. For consistency in the notation we define $\mathrm{syz}_{-1}(\mathbf{G}(G,q)) = \{0\}$ and the map

$$\psi_{-1} : \mathfrak{S}_1(G,q) \hookrightarrow \{0\}$$

sends the canonical connected 1-flag $V(G)$ to 0. The proof is by induction on $k \geq 0$.

Base case. For $k = 0$ the result is proved in [CRS02, Theorem 14]. Here $\mathbf{G}_0(G,q) = \mathbf{G}(G,q)$ and $<_0$ is $<$, and

$$\begin{aligned} \psi_0 : \mathfrak{S}_2(G,q) &\hookrightarrow \mathrm{syz}_0(\mathbf{G}(G,q)) = I_G \\ (U_1 \subsetneq U_2) &\mapsto (\mathbf{x}^{D(U_2 \setminus U_1, U_1)} - \mathbf{x}^{D(U_1, U_2 \setminus U_1)})[0] \end{aligned}$$

and $\mathrm{LM}(\psi_k(\mathcal{U})) = \mathbf{x}^{D(U_2 \setminus U_1, U_1)}[0]$.

Induction hypothesis. Now let $k > 0$ and assume that there exists a bijection

$$\psi_{k-1} : \mathfrak{S}_{k+1}(G,q) \to \mathbf{G}_{k-1}(G,q) \subseteq \mathrm{syz}_{k-1}(\mathbf{G}(G,q))$$

such that $\mathbf{G}_{k-1}(G,q)$ forms a minimal homogeneous Gröbner basis of $\mathrm{syz}_{k-1}(\mathbf{G}(G,q))$ with respect to $<_{k-1}$), and (5) for the leading monomials holds.

Via the bijection ψ_{k-1}, the set $\mathbf{G}_{k-1}(G,q)$ inherits a total ordering $\prec'_{k-1}$ from the total ordering $\prec_{k+1}$ on $\mathfrak{S}_{k+1}(G,q)$, i.e.

$$f \prec'_{k-1} h \quad \text{in} \quad \mathbf{G}_{k-1}(G,q) \quad \Leftrightarrow \quad \psi_{k-1}^{-1}(f) \prec_{k+1} \psi_{k-1}^{-1}(h) \quad \text{in} \quad \mathfrak{S}_{k+1}(G,q).$$

Inductive step. Given $\mathcal{U} \in \mathfrak{S}_{k+2}(G,q)$ let $\mathcal{U}^{(1)}$ and $\mathcal{U}^{(2)}$ be as defined in Definition 3.9. We define

$$\psi_k : \mathfrak{S}_{k+2}(G,q) \to \mathrm{syz}_k(\mathbf{G}(G,q)) \tag{6}$$

$$\mathcal{U} \mapsto s(\psi_{k-1}(\mathcal{U}^{(1)}), \psi_{k-1}(\mathcal{U}^{(2)}))\ .$$

In the following $\mathcal{U}, \mathcal{V} \in \mathfrak{S}_{k+2}(G,q)$ are of the form

$$U_1 \subsetneq U_2 \subsetneq \cdots \subsetneq V(G)$$

$$V_1 \subsetneq V_2 \subsetneq \cdots \subsetneq V(G)\ .$$

The result is follows from a series of claims.

Claim 1. ψ_k is a well-defined and $\mathbf{G}_k(G,q) := \mathrm{Image}(\psi_k)$ consists of homogeneous elements.

Since $\psi_{k-1}(\mathcal{U}^{(1)})$ and $\psi_{k-1}(\mathcal{U}^{(2)})$ are homogeneous by induction hypothesis, it follows that $s(\psi_{k-1}(\mathcal{U}^{(1)}), \psi_{k-1}(\mathcal{U}^{(2)}))$ is also homogeneous.

Claim 2. $\mathrm{LM}(\psi_k(\mathcal{U})) = \mathbf{x}^{D(U_2\backslash U_1, U_1)}[\psi_{k-1}(\mathcal{U}^{(1)})]$.

It suffices to show that $D(U_2\backslash U_1, U_1) = \max(\alpha, \beta) - \alpha$ where

$$\mathrm{LM}(\psi_{k-1}(\mathcal{U}^{(1)})) = \mathbf{x}^{\alpha}[\psi_{k-2}(\mathcal{U}^{(1,1)})] \quad , \quad \mathrm{LM}(\psi_{k-1}(\mathcal{U}^{(2)})) = \mathbf{x}^{\beta}[\psi_{k-2}(\mathcal{U}^{(2,1)})]\ .$$

Claim 3. ψ_k is injective.

If $\mathcal{U}, \mathcal{V} \in \mathfrak{S}_{k+2}(G,q)$ be such that $\psi_k(\mathcal{U}) = \psi_k(\mathcal{V})$ then their leading terms are equal :

$$\mathbf{x}^{D(U_2\backslash U_1, U_1)}[\psi_{k-1}(\mathcal{U}^{(1)})] = \mathbf{x}^{D(V_2\backslash V_1, V_1)}[\psi_{k-1}(\mathcal{V}^{(1)})]\ .$$

Therefore $\psi_{k-1}(\mathcal{U}^{(1)}) = \psi_{k-1}(\mathcal{V}^{(1)})$ and $D(U_2\backslash U_1, U_1) = D(V_2\backslash V_1, V_1)$. By induction hypothesis ψ_{k-1} is injective which implies $\mathcal{U}^{(1)} = \mathcal{V}^{(1)}$ and $D(U_2\backslash U_1, U_1) = D(V_2\backslash V_1, V_1)$. Therefore $U_1 = V_1$ and $\mathcal{U} = \mathcal{V}$.

The following claim (proved in [MS12]) will finish the inductive step.

Claim 4. $\mathrm{Image}(\psi_k)$ forms a minimal homogeneous Gröbner basis of $\mathrm{syz}_k(\mathbf{G}(G,q))$ with respect to $<_k$ obtained from $<_{k-1}$ according to (2).

These all together show that $\mathrm{Image}(\psi_k) \subseteq \mathcal{S}(\mathbf{G}_{k-1}(G,q))$. In order to show the reverse inclusion by Theorem 2.2 it remains to show that

(I) $0 \notin \mathrm{Image}(\psi_k)$.

(II) For any element $s(f,h) \in \mathcal{S}(\mathbf{G}_{k-1}(G,q))$ there exists an element $g \in \mathrm{Image}(\psi_k)$ such that $\mathrm{LM}(g) \mid \mathrm{LM}(s(f,h))$.

(III) For any two elements $g, g' \in \mathrm{Image}(\psi_k)$, if $\mathrm{LM}(g) \mid \mathrm{LM}(g')$ then $g = g'$.

Claim 5. For $\mathcal{U} \in \mathfrak{S}_{k+2}(G,q)$ we have $\psi_k(\mathcal{U}) = \sum_{\mathcal{W}\in\mathfrak{S}_{k+1}(G,q)} c(\mathcal{U},\mathcal{W})\mathbf{x}^{\theta(\mathcal{U},\mathcal{W})}[\psi_{k-1}(\mathcal{W})]$ where $c(\mathcal{U},\mathcal{W}) \in \{-1,0,1\}$ and $\theta(\mathcal{U},\mathcal{W}) = D(U_i\backslash U_{i-1}, U_j\backslash U_{j-1})$ if $\mathcal{W}$ differs from $\mathcal{U}$ by merging $U_i\backslash U_{i-1}$ and $U_j\backslash U_{j-1}$ for some i,j.

Note that this proves (III) which is equivalent to the minimality of the resolution. □

From Theorem 4.1 we obtain the following important corollaries.

Corollary 4.2 *The Betti numbers of the ideals I_G and $\mathrm{in}(I_G)$ are independent of the characteristic of the base field K.*

Corollary 4.3 *For all $i \geq 0$, $\beta_i(R/I_G) = \beta_i(R/\mathrm{in}(I_G)) = |\mathfrak{S}_{i+1}(G,q)| = |\mathfrak{E}_{i+1}(G,q)|$.*

Let $\mathsf{A} = \mathbb{Z}$ or $\mathsf{A} = \mathrm{Pic}(G)$. Recall that I_G and $\mathrm{in}(I_G)$ are graded (or homogeneous) with respect to the $\mathbb{Z}$ and $\mathrm{Pic}(G)$ gradings. One can also read the graded Betti numbers from Theorem 4.1.

Corollary 4.4 *For all $i \geq 0$ and $\mathsf{j} \in \mathsf{A}$ we have $\beta_{i,\mathsf{j}} = |\mathfrak{S}_{i+1,\mathsf{j}}(G,q)|$ where $\mathfrak{S}_{k,\mathsf{j}}(G,q) = \{\mathcal{U} \in \mathfrak{S}_k(G,q) : \deg_{\mathsf{A}}(\mathbf{x}^{D(\mathcal{U})}) = \mathsf{j}\}$.*

We conclude this section with some examples.

Example 4.5 It follows from above descriptions that $\beta_{n-1}(R/I_G) = \beta_{n-1,m}(R/I_G)$ is equal to the number of acyclic orientations of G with unique source.

Example 4.6 Let G be the complete graph K_n on n vertices. Then $\beta_{k-1}(R/I_G) = |\mathfrak{S}_k(G,q)| = (k-1)!\,S(n,k)$ where $S(n,k)$ denotes the Stirling number of the second kind (i.e. the number of ways to partition a set of n elements into k nonempty subsets).

Example 4.7 Let G be a tree on n vertices. Then $\beta_{k-1}(R/I_G) = |\mathfrak{S}_k(G,q)| = \binom{n-1}{k-1}$.

Example 4.8 For the cycle C_n on n vertices and $k \geq 2$ we have $\beta_{k-1}(R/I_{C_n}) = |\mathfrak{S}_k(C_n,q)| = (k-1) \times \binom{n}{k}$.

5 Relation to maximal reduced divisors

Recall the definition of reduced divisors.

Definition 5.1 *Let (Γ, v_0) be a pointed graph. A divisor $D \in \mathrm{Div}(\Gamma)$ is called* v_0-reduced *if it satisfies the following two conditions :*

(i) $D(v) \geq 0$ for all $v \in V(\Gamma)\backslash\{v_0\}$.

(ii) For every non-empty subset $A \subseteq V(\Gamma)\backslash\{v_0\}$, there exists a vertex $v \in A$ such that $D(v) < \mathrm{outdeg}_A(v)$.

These divisors arise precisely from the normal forms with respect to the Gröbner basis given in Theorem 4.1. There is a well-known algorithm due to Dhar for checking whether a given divisor is reduced (see, e.g., [BS13] and references therein).

Lemma 5.2 *For $\mathcal{U} \in \mathfrak{S}_k(G,q)$, $\phi_*(D(\mathcal{U})) = E + \mathbf{1}$, where E is a maximal $(\phi(U_1))$-reduced divisor and $\mathbf{1}$ is the all-one divisor.*

Since different acyclic orientations with unique source at q' give rise to inequivalent q'-reduced divisors we deduce that if $\mathcal{U}, \mathcal{V} \in \mathfrak{S}_k(G,q)$ and the graphs $G_{/\mathcal{U}}$ and $G_{/\mathcal{V}}$ coincide, then $\phi_*(D(\mathcal{U})) - \mathbf{1}$ and $\phi_*(D(\mathcal{V})) - \mathbf{1}$ are two inequivalent maximal reduced divisors. These observations lead to the following formula for Betti numbers which was conjectured in [PPW11] for I_G :

$$\begin{aligned}\beta_i &= \sum_{G_{/\mathcal{U}}} |\{D : D \text{ is a maximal } v_0\text{-reduced divisors on } G_{/\mathcal{U}}\}| \\ &= \sum_{G_{/\mathcal{U}}} |\{\text{acyclic orientations of } G_{/\mathcal{U}} \text{ with unique source at } v_0\}|\end{aligned}$$

where the sum is over all *distinct* contracted graphs $G_{/\mathcal{U}}$ as $\mathcal{U}$ varies in $\mathfrak{S}_{i+1}(G,q)$, and v_0 is an arbitrary vertex of $G_{/\mathcal{U}}$.

Here is another connection with reduced divisors. Hochster's formula for computing the Betti numbers topologically (see, e.g., [MS05, Theorem 9.2]), when applied to I_G and the "nice" grading by $\mathrm{Pic}(G)$, says that for each $\mathrm{j} \in \mathrm{Pic}(G)$ the graded Betti number $\beta_{i,\mathrm{j}}(R/I_G)$ is the dimension of the i^{th} reduced homology of the simplicial complex $\Delta_{\mathrm{j}} = \{\mathrm{supp}(E) : 0 \leq E \leq D' \in |\,\mathrm{j}\,|\}$ where $|\,\mathrm{j}\,|$ denotes the linear system of $\mathrm{j} \in \mathrm{Pic}(G)$.

Remark 5.3

(i) For $\mathrm{j} \in \mathrm{Pic}(G)$, we have $\beta_{n-1,\mathrm{j}}(R/I_G) = 1$ if and only if $\mathrm{j} \sim E + \mathbf{1}$ where E is a maximal q-reduced divisor.

(ii) One can use Corollary 4.3 to read all dimensions of the reduced homologies of Δ_{j}. Although we now know all the Betti numbers, giving an explicit bijection between connected flags and the bases of the reduced homologies of Δ_{j} is an intriguing problem. In a recent work, Horia Mania in [Man12] studies the number of connected components of Δ_{j}.

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