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Dual Equivalence Graphs Revisited with Applications to LLT and Macdonald Polynomials

Austin Roberts†

Department of Mathematics, University of Washington, Seattle, Washington, USA

Abstract. In 2007 Sami Assaf introduced dual equivalence graphs as a method for demonstrating that a quasisymmetric function is Schur positive. The method involves the creation of a graph whose vertices are weighted by Ira Gessel's fundamental quasisymmetric functions so that the sum of the weights of a connected component is a single Schur function. In this paper, we improve on Assaf's axiomatization of such graphs, giving locally testable criteria that are more easily verified by computers. We then demonstrate the utility of this result by giving explicit Schur expansions for a family of Lascoux-Leclerc-Thibon polynomials. This family properly contains the previously known case of polynomials indexed by two skew shapes, as was described in a 1995 paper by Christophe Carré and Bernard Leclerc. As an immediate corollary, we gain an explicit Schur expansion for a family of modified Macdonald polynomials in terms of Yamanouchi words. This family includes all polynomials indexed by shapes with less than four cells in the first row and strictly less than three cells in the second row, a slight improvement over the known two column case described in 2005 by James Haglund, Mark Haiman, and Nick Loehr.

Résumé. En 2007, Sami Assaf a introduit la mé thode des graphes d'é quivalence duale pour dé montrer qu'une fonction quasisymmé trique est Schur-positive. La mé thode né cessite la cré ation d'un graphe dont les sommets sont pondé ré s par les fonctions quasisymmé triques fondamentales d'Ira Gessel tel que la somme des poids d'une composante connexe soit une unique fonction de Schur. Dans cet article, nous amé liorons l'axiomatisation d'Assaf pour ces graphes, et nous obtenons des critè res locaux qui sont plus facilement vé rifié s par ordinateur. Puis nous appliquons ces techniques pour pré senter des dé veloppements explicites en fonctions de Schur d'une famille de polynô mes de Lascoux-Leclerc-Thibon. Cette famille contient strictement le cas des polynô mes indexé s par deux formes gauches, qui a é té dé crit dans un article en 1995 de Christophe Carré et Bernard Leclerc. Comme corollaire immé diat, nous obtenons un dé veloppement explicite en fonctions de Schur d'une famille de polynô mes de Macdonald modifié s, exprimé e au moyen de mots de Yamanouchi. Cette famille inclut tous les polynô mes indexé s par des formes de moins de quatre cellules dans la premiè re ligne et strictement moins de trois cellules dans la deuxiè me ligne, ce qui est une lé gè re amé lioration par rapport au cas connu de deux colonnes dé crit en 2005 par James Haglund, Mark Haiman, et Nick Loehr.

Keywords: Dual equivalence graph, LLT polynomial, Macdonald polynomial, Schur expansion, quasisymmetric function

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1 Introduction

Dual equivalence was developed and applied by Mark Haiman in (Haiman, 1992) as an extension of work done by Donald Knuth in (Knuth, 1970). Sami Assaf then introduced the theory of dual equivalence graphs in her Ph.D. dissertation Assaf (2007) and subsequent preprint Assaf (2011). In these papers, she is able to associate a number of symmetric functions to dual equivalence graphs and each component of a dual equivalence graph to a Schur function, thus demonstrating Schur positivity. More recently, variations of dual equivalence graphs are given for k-Schur functions in Assaf and Billey (2012) and for the product of a Schubert polynomial with a Schur polynomial in Assaf et al. (2012).

A key connection between dual equivalence graphs and symmetric functions is the ring of quasisymmetric functions. The quasisymmetric functions were introduced by Ira Gessel (1984) as part of his work on P-partitions. Currently there are a number of functions that are easily expressible in terms of Gessel's *fundamental quasisymmetric functions* that are not easily expressed in terms of Schur functions. For example, such an expansion for plethysms is described in Loehr and Warrington (2012), for Lascoux-Leclerc-Thibon (LLT) polynomials in Haglund et al. (2005b), and for Macdonald polynomials in Haglund et al. (2005a). An expressed goal of developing the theory of dual equivalence graphs is to create a tool for turning such quasisymmetric expansions into explicit Schur expansions.

Previously, dual equivalence graphs were defined by five *dual equivalence axioms* that are locally testable and one that is not. In attempting to apply the theory of dual equivalence graphs, it is often a challenge to demonstrate that this nonlocal axiom is satisfied. The main results of this paper is to give an equivalent definition using only local conditions, as is stated in Theorem 2.9.

The paper concludes by applying the above result to LLT polynomials. LLT polynomials were first introduced in Lascoux et al. (1997) as a q-analogue to products of Schur functions and were later given a description in terms of tuples of skew tableaux in Haglund et al. (2005b). LLT polynomials can, in turn, be used to give an explicit combinatorial description of modified Macdonald polynomials by using the results of Haglund et al. (2005a). First introduced in Macdonald (1988), Macdonald polynomials are often defined as the set of q, t-symmetric functions that satisfy certain orthogonality and triangularity conditions, as is well described in Macdonald (1995). Part of the importance of Macdonald polynomials derives from the fact that they specialize to a wide array of well known functions, including Hall-Littlewood polynomials and Jack polynomials (see Macdonald (1995) for details). In Haiman (2001), Mark Haiman used geometric and representation-theoretic techniques to prove that Macdonald polynomials are Schur positive.

In some cases, nice Schur expansions for LLT and Macdonald polynomials are already known. In particular, the set of LLT polynomials indexed by two skew shapes was described in Carré and Leclerc (1995) and van Leeuwen (2000), and modified Macdonald polynomials indexed by shapes with strictly less than three columns was described in Haglund et al. (2005a) (which in turn drew on the earlier work in Carré and Leclerc (1995), van Leeuwen (2000)). The first combinatorial description of the two column case was given in Fishel (1995), but others were subsequently given in Zabrocki (1998), Lapointe and Morse (2003), and Assaf (2008/09).

This paper is broken into sections as follows. Section 2 is dedicated giving a new axiomatization for dual equivalence graphs in Theorem 2.9. Section 3 applies the results of Section 2 to LLT polynomials and Macdonald polynomials. The graph structure given to LLT polynomials in Assaf (2011) is reviewed. Theorem 3.9 states that the set of LLT polynomials corresponding to said graphs have a Schur expansion indexed by standardized Yamanouchi words. This set strictly contains the set of LLT polynomials indexed by two skew shapes. Corollary 3.10 then gives a Schur expansion for modified Macdonald polynomials indexed by partition shapes with strictly less than four boxes in the first row and strictly less than three boxes in the second row. For a fuller account of all of these results, see Roberts (2013).

2 The Structure of Dual Equivalence Graphs

2.1 Preliminaries

In this section we provide the necessary definitions and results from Assaf (2011). We begin by recalling Mark Haiman's dual to the fundamental Knuth equivalences.

Definition 2.1 Given a permutation in S_n expressed in one-line notation, define an *elementary dual equivalence* as an involution d_i that interchanges the values $i-1, i$, and $i+1$ as

$$\begin{aligned} d_i(\ldots i \ldots i-1 \ldots i+1 \ldots) &= (\ldots i+1 \ldots i-1 \ldots i \ldots), \\ d_i(\ldots i-1 \ldots i+1 \ldots i \ldots) &= (\ldots i \ldots i+1 \ldots i-1 \ldots), \end{aligned} \tag{2.1}$$

and acts as the identity if i occurs between $i-1$ and $i+1$. Two words are $dual\ equivalent$ if one may be transformed into the other by successive elementary dual equivalences.

For example, 21345 is dual equivalent to 41235 because $d_3(d_2(21345)) = d_3(31245) = 41235$.

We may also let d_i act on the entries of a tableau via the row reading word. It is simple to check that the result is again a tableau of the same shape. The transitivity of this action is described in the next theorem.

Theorem 2.2 ((Haiman, 1992, Prop. 2.4)) *Two standard Young tableaux on partition shapes are dual equivalent if and only if they have the same shape.*

The signature of a permutation is a string of 1's and -1's, or +'s and $-$'s for short, where there is a + in the i^{th} position if and only if i comes before $i+1$ in one-line notation. We may then define the signature of a tableau T, denoted $\sigma(T)$, as the signature of the row reading word of T. If we wish to be explicit about this definition of σ, we will refer to the signature function as being *given by inverse descents*.

By definition, d_i is an involution, and so we define a graph on standard Young tableaux by letting each nontrivial orbit of d_i define an edge colored by i. By Theorem 2.2, the graph on SYT(n) with edges labeled by $1 < i < n$ has connected components with vertices in SYT(λ) for each $\lambda \vdash n$. We may further label each vertex with its signature to create a *standard dual equivalence graph* that we will denote $\mathcal{G}_\lambda$. See Figure 1 for examples.

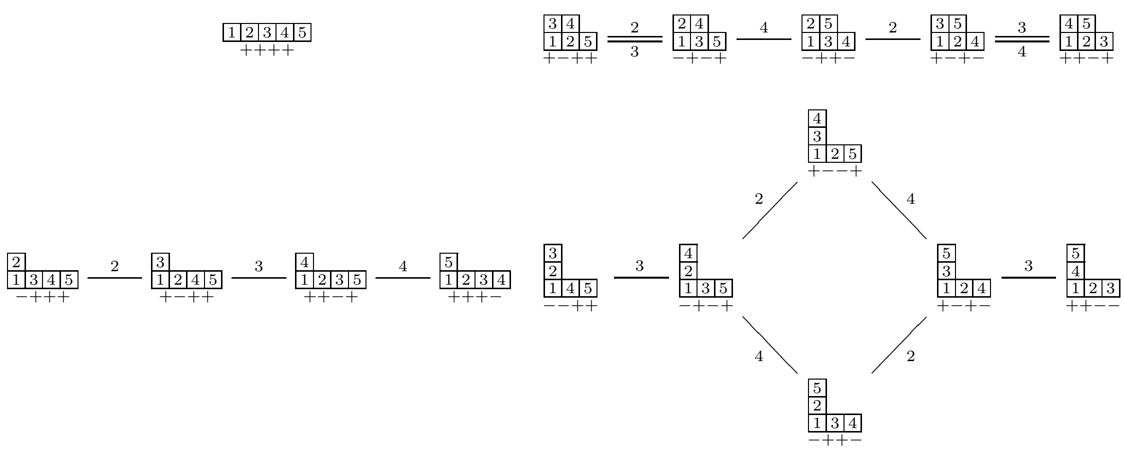

Fig. 1: The standard dual equivalence graphs on partitions of 5 up to conjugation.

Here, $\mathcal{G}_\lambda$ is an example of the following broader class of graphs.

Definition 2.3 An *edge colored graph* consists of the following data:

1. *a finite vertex set V,*
2. *a collection E_i of unordered pairs of distinct vertices in V for each $i \in \{m+1, \ldots, n-1\}$, where m and n are positive integers.*

A *signed colored graph* is an edge colored graph with the following additional data:

3. *a signature function $\sigma : V \to \{\pm 1\}^{N-1}$, for some positive integer $N \geq n$.*

We denote a signed colored graph by $\mathcal{G} = (V, \sigma, E_{m+1} \cup \cdots \cup E_{n-1})$ or simply $\mathcal{G} = (V, \sigma, E)$. If a signed colored graph has $m = 1$, then it is said to have *type* (n, N) and is termed an (n, N)-signed colored graph.

We may also restrict signed colored graphs. If $\mathcal{G}$ is an (n, N)-signed colored graph, $M \leq N$, and $m \leq n$, then the (m, M)*-restriction* of $\mathcal{G}$ is the result of excluding E_i for $i \geq m$ and projecting each signature onto its first $M - 1$ coordinates.

In order to present structural results about signed colored graphs, we first need to define isomorphisms.

Definition 2.4 A map $\phi : \mathcal{G} \to \mathcal{H}$ between edge colored graphs $\mathcal{G} = (V, E_{m+1} \cup \ldots \cup E_{n-1})$ and $\mathcal{H} = (V', E'_{m+1} \cup \ldots \cup E'_{n-1})$ is called a $morphism$ if it preserves i-edges. That is, $\{v, w\} \in E_i$ implies $\{\phi(v), \phi(w)\} \in E'_i$ for all $v, w \in V$ and all $m < i < n$.

A map $\phi : \mathcal{G} \to \mathcal{H}$ between signed colored graphs $\mathcal{G} = (V, \sigma, E_{m+1} \cup \ldots \cup E_{n-1})$ and $\mathcal{H} = (V', \sigma', E'_{m+1} \cup \ldots \cup E'_{n-1})$ is called a *morphism* if it is a morphism of edge colored graphs that also preserves signatures. That is, $\sigma'(\phi(v)) = \sigma(v)$.

In both cases, a morphism is an $isomorphism$ if it admits an inverse morphism.

Notice that in a standard dual equivalence graph, a vertex v is included in an i-edge if and only if $\sigma(v)_{i-1} = -\sigma(v)_i$, motivating the following definition.

Definition 2.5 Let $\mathcal{G} = (V, \sigma, E)$ be a signed colored graph. We say that $w \in V$ *admits an i-neighbor* if $\sigma(w)_{i-1} = -\sigma(w)_i$.

We are now ready to present an axiomatization of the structure inherent in a standard dual equivalence graph.

Definition 2.6 A signed colored graph $\mathcal{G} = (V, \sigma, E_{m+1} \cup \ldots \cup E_{n-1})$ is a *dual equivalence graph* if the following hold:

(ax1): For $m < i < n$, each E_i is a complete matching on the vertices of V that admit an i-neighbor.
(ax2): If $\{v, w\} \in E_i$, then $\sigma(v)_i = -\sigma(w)_i, \sigma(v)_{i-1} = -\sigma(w)_{i-1}$, and $\sigma(v)_h = \sigma(w)_h$ for all $h < i - 2$ and all $h > i + 1$.
(ax3): For $\{v, w\} \in E_i$, if σ_{i-2} is defined, then v or w (or both) admits an $(i - 1)$-neighbor, and if σ_{i+1} is defined, then v or w (or both) admits an $(i + 1)$-neighbor.
(ax4): For all $m+1 < i < n$, any component of the edge colored graph $(V, E_{i-2} \cup E_{i-1} \cup E_i)$ is isomorphic to a component of the restriction of some $\mathcal{G}_\lambda = (V', \sigma', E')$ to $(V', E'_{i-2} \cup E'_{i-1} \cup E'_i)$, where E_{i-2} is omitted if $i = m + 2$ (see Figure 2).
(ax5): For all $1 < i, j < n$ such that $|i - j| > 2$, if $\{v, w\} \in E_i$ and $\{w, x\} \in E_j$, then there exists $y \in V$ such that $\{v, y\} \in E_j$ and $\{x, y\} \in E_i$.
(ax6): For all $m < i < n$, any two vertices of a connected component of $(V, \sigma, E_{m+1} \cup \cdots \cup E_i)$ may be connected by some path crossing at most one E_i edge.

A dual equivalence graph that is also an (n, N)-signed colored graph is said to have *type* (n, N) and is termed an (n, N)-*dual equivalence graph*.

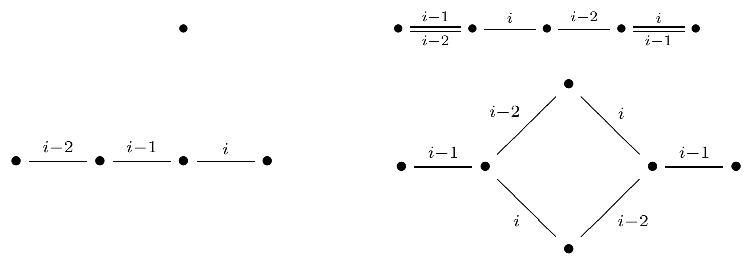

Fig. 2: Allowable $E_{i-2} \cup E_{i-1} \cup E_i$ components of Axiom 4.

The next theorem links the definitions of dual equivalence graphs and standard dual equivalence graphs.

Theorem 2.7 ((Assaf, 2011, Theorems 3.5 and 3.9)) *A connected component of an* (n, n)*-signed colored graph is a dual equivalence graph if and only if it is isomorphic to a unique* $\mathcal{G}_\lambda$.

2.2 Local Conditions for Axiom 6

Axiom 6 is the only dual equivalence axiom that cannot be tested locally, and in practice it is often a barrier to applying the theory of dual equivalence graphs. In this section, we show how to replace Axiom 6 with a locally testable axiom.

Definition 2.8 A signed colored graph $\mathcal{G} = (V, \sigma, E_{m+1} \cup \ldots \cup E_{n-1})$ is said to obey Axiom 4^+ if for all $m + 1 < i < n$, any component of the edge colored graph $(V, E_{i-3} \cup E_{i-2} \cup E_{i-1} \cup E_i)$ is isomorphic to a component of the restriction of some $\mathcal{G}_\lambda = (V', \sigma', E')$ to $(V', E'_{i-3} \cup E'_{i-2} \cup E'_{i-1} \cup E'_i)$, where E_{i-3}, E_{i-2}, or E_{i-1} is omitted if $i \leq m + 3, i \leq m + 2$, or $i = m + 1$, respectively.

Notice that Axiom 4^+ is just an extension of Axiom 4 to components with 4 consecutive edge colors. The next theorem states that this extension to Axiom 4^+ allows for the omission of Axiom 6 in the dual equivalence axioms.

Theorem 2.9 *A signed colored graph satisfies Axioms 1, 2, 3,* 4^+*, and 5 if and only if it is a dual equivalence graph.*

Proof Sketch: The crux of the proof is to show that Axioms 1, 2, 3, 4^+, and 5 imply Axiom 6. We provide a sketch of this argument here, proceeding by induction on n. For $n \leq 6$ the result is largely a consequence of the definition of Axiom 4^+. We may then assume the result when $n - 1 \leq 6$ and prove that an arbitrary (n, n)-signed colored graph $\mathcal{G}$ satisfying Axioms 1, 2, 3, 4^+, and 5 must also satisfy Axiom 6.

It follows from the proof of Theorem 2.7 that $\mathcal{G}$ admits a morphism onto some standard dual equivalence graph $\mathcal{G}_\lambda$ and that each $(n - 1, n - 1)$-component of $\mathcal{G}$ is connected in $\mathcal{G}$ to exactly one $(n - 1, n - 1)$-components of each possible isomorphism type. We may then show that every $(n - 1, n - 1)$-component of $\mathcal{G}$ has a unique isomorphism type, establishing Axiom 6.

To prove this last point, we consider two cases. In the first case, we assume that λ is not a staircase. In the second case, we assume that λ has at least four Northeast corners. In both cases, induction is used to provide the required connectivity result. Finally, the only cases that are staircases but have less than four Northeast corners have at most six cells, and so are covered in our base case. $\square$

Remark 2.10 We may readily classify the set of edge colored graphs described in Definition 2.8, i.e., the set of edge colored graphs that arise as components of the restriction of some $\mathcal{G}_\lambda = (V', \sigma', E')$ to $(V', E'_{i-3} \cup E'_{i-2} \cup E'_{i-1} \cup E'_i)$. Each such edge colored graph is the result of restricting some $\mathcal{G}_\lambda = (V, \sigma, E)$ to (V, E) and adding h to each edge label, where $\lambda \vdash 6$ and h is some nonnegative integer.

Let $\mathcal{F}$ be the set of edge colored graphs with edge sets $E_{i-3} \cup \ldots \cup E_i$ that satisfy Axioms 4 and 5 but not Axiom 6. A canonical graph in $\mathcal{F}$ is presented in Figure 3. The following corollary reformulates Theorem 2.9 in terms of $\mathcal{F}$.

Corollary 2.11 *Let $\mathcal{G} = (V, \sigma, E_{m+1} \cup \ldots \cup E_{n-1})$ be a signed colored graph satisfying Axioms 1, 2, 3, 4, and 5. Then $\mathcal{G}$ is a dual equivalence graph if and only if for all $m + 4 < i < n$, the restriction of $\mathcal{G}$ to the edge colored graph $(V, E_{i-4} \cup \ldots \cup E_i)$ has no components isomorphic to an element of $\mathcal{F}$.*

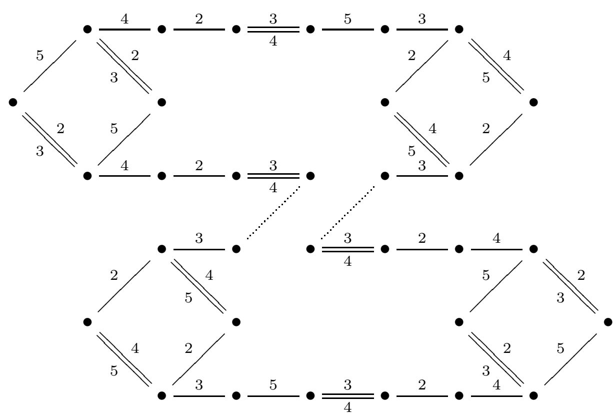

Fig. 3: A generic graph in $\mathcal{F}$ with edge labels in $\{2, 3, 4, 5\}$.

Remark 2.12 For any edge colored graph in $\mathcal{F}$, every vertex shares an edge with at least two other vertices. We may then give yet another characterization of dual equivalence graphs. Let $\mathcal{G} = (V, \sigma, E_{m+1} \cup \ldots \cup E_{n-1})$ be a signed colored graph obeying Axioms 1, 2, 3, 4, and 5. Choose $\mathcal{C}$ to be any component of the restriction of $\mathcal{G}$ to the edge colored graph $(V, E_{i-3} \cup E_{i-2} \cup E_{i-1} \cup E_i)$ such that $m + 3 < i < n$ and the vertices of $\mathcal{C}$ all have at least two adjacent vertices in $\mathcal{C}$. Then $\mathcal{G}$ is a dual equivalence graph if and only if $\mathcal{C}$ is not in $\mathcal{F}$ for any choice of $\mathcal{C}$. This characterization of dual equivalence graphs is used in the computer verification of Theorem 3.7.

3 LLT and Macdonald Polynomials

In this section, we demonstrate the utility of the results in Section 2 by applying them to LLT Polynomials.

3.1 Symmetric Functions

We begin by recalling the definitions of the necessary symmetric functions. Crucial to our definitions will be the definition of the fundamental quasisymmetric function.

Definition 3.1 Given any signature $\sigma \in \{\pm 1\}^{n-1}$, define the *fundamental quasisymmetric function* $F_\sigma(X) \in \mathbb{Z}[x_1, x_2, \ldots]$ by

$$F_\sigma(X) := \sum_{\substack{i_1 \leq \ldots \leq i_n \\ i_j = i_{j+1} \Rightarrow \sigma_j = +1}} x_{i_1} \cdots x_{i_n}.$$

We take the unorthodox approach of using a result by Gessel (1984) to give our next definition.

Definition 3.2 Gessel (1984) Given any skew shape λ/ρ, define

$$s_{\lambda/\rho}(X) := \sum_{T \in \mathrm{SYT}(\lambda/\rho)} F_{\sigma(T)}(X), \tag{3.1}$$

where $s_{\lambda/\rho}$ is termed a Schur function if λ/ρ is a straight shape and a skew Schur function in general.

Definition 3.2 and Theorem 2.2 determine the connection between Schur functions and dual equivalence graphs as highlighted in (Assaf, 2011, Cor. 3.10). Given any standard dual equivalence graph $\mathcal{G}_\lambda = (V, \sigma, E)$,

$$\sum_{v \in V} F_{\sigma(v)} = s_\lambda. \tag{3.2}$$

Given a a diagram of shape λ/ρ in French notation, the *content* of cell x, denoted $c(x)$, is $j - i$, where j is the column of x and i is the row of x in Cartesian coordinates. Here, the lower left corner of λ is assumed to be at the origin. Given a k-tuple of skew shapes $\boldsymbol{\nu} = (\nu^{(0)}, \ldots, \nu^{(k-1)})$, we write $|\boldsymbol{\nu}| = n$ if $\sum_{i=0}^{k-1} |\nu^{(i)}| = n$. A standard filling $\mathbf{T} = (T^{(0)}, \ldots, T^{(k-1)})$ of $\boldsymbol{\nu}$ is a bijective filling of the diagram of $\boldsymbol{\nu}$ with entries in $[n]$ such that each $T^{(i)}$ is strictly increasing up columns and across rows from left to right. Denote the set of standard fillings of $\boldsymbol{\nu}$ as $\mathrm{SYT}(\boldsymbol{\nu})$. Define the *shifted content* of cell x in $\nu^{(i)}$ as,

$$\tilde{c}(x) = k \cdot c(x) + i, \tag{3.3}$$

where $c(x)$ is the content of x in $\nu^{(i)}$. The *shifted content word* of $\mathbf{T}$ is defined as the word retrieved from reading off the values in the cells from lowest shifted content to highest, reading northeast along diagonals of constant shifted content. We may then define $\sigma(\mathbf{T})$ as the signature of the shifted content word of $\mathbf{T}$.

Letting $\mathbf{T}(x)$ denote the entry in cell x, the set of k-*inversions of* $\mathbf{T}$ is

$$\mathrm{Inv}_k(\mathbf{T}) := \{(x, y) \mid k > \tilde{c}(y) - \tilde{c}(x) > 0 \text{ and } \mathbf{T}(x) > \mathbf{T}(y)\}. \tag{3.4}$$

The k-*inversion number of* $\mathbf{T}$ is defined as

$$\mathrm{inv}_k(\mathbf{T}) := |\mathrm{Inv}_k(\mathbf{T})|. \tag{3.5}$$

Now define the set of LLT polynomials by

$$\widetilde{G}_{\boldsymbol{\nu}}(X; q) := \sum_{\mathbf{T} \in \mathrm{SYT}(\boldsymbol{\nu})} q^{\mathrm{inv}_k(\mathbf{T})} F_{\sigma(\mathbf{T})}(X). \tag{3.6}$$

We now move on to the definition of the modified Macdonald polynomials $\widetilde{H}_{\mu/\rho}(X; q, t)$. We will use (Haglund et al., 2005a, Theorem 2.2) to give a strictly combinatorial definition, using the statistics "inv", "maj", and "a" from this paper without defining them here. We will, however, need one new definition.

Given any skew shape μ/ρ with each cell represented by a pair (i, j) in Cartesian coordinates, let $\mathrm{TR}(\mu/\rho)$ be the set of tuples of ribbons $\boldsymbol{\nu} = (\nu^{(0)}, \ldots, \nu^{(k-1)})$, such that $\nu^{(i)}$ has a cell with content j if

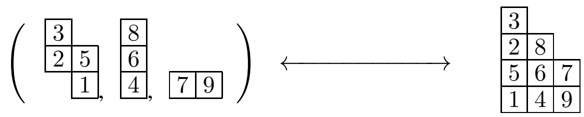

Fig. 4: An example of the bijection between standard fillings of shapes in TR(μ/ρ) and bijective fillings of μ/ρ. At left, the cells labeled 1, 4, and 9 have content 0 in their respective ribbons.

and only if $(i, -j)$ is a cell in μ/ρ. There is then a bijection between standard fillings of shapes in TR(μ/ρ) and bijective fillings of μ/ρ given by turning each ribbon into a column of μ/ρ as demonstrated in Figure 4.

We are now able to define the modified Macdonald polynomials and show their relationship with LLT polynomials.

$$\widetilde{H}_{\mu/\rho}(X;q,t) := \sum_{\substack{\boldsymbol{\nu}\in \mathrm{TR}(\mu/\rho)\\ \mathbf{T}\in \mathrm{SYT}(\boldsymbol{\nu})}} q^{\mathrm{inv}(\mathbf{T})} t^{\mathrm{maj}(\mathbf{T})} F_{\sigma(\mathbf{T})} = \sum_{\boldsymbol{\nu}\in \mathrm{TR}(\mu/\rho)} q^{-a(\boldsymbol{\nu})} t^{\mathrm{maj}(\boldsymbol{\nu})} \widetilde{G}_{\boldsymbol{\nu}}(X;q) \tag{3.7}$$

By using this definition, results about LLT polynomials can be easily translated into results about Macdonald polynomials.

3.2 LLT graphs

We follow Assaf (2011) in defining an involution that will provide the edge sets of a signed colored graph. In this section, $\boldsymbol{\nu}$ will always denote a k-tuple of skew shapes whose sizes sum to $|\boldsymbol{\nu}| = n$. Also, w will always denote a permutation in S_n.

Let the involution $\tilde{d}_i : S_n \to S_n$ act by permuting the entries $i-1, i$, and $i+1$ as defined by,

$$\begin{aligned} \tilde{d}_i(\ldots i \ldots i-1 \ldots i+1 \ldots) &= (\ldots i-1 \ldots i+1 \ldots i \ldots),\\ \tilde{d}_i(\ldots i \ldots i+1 \ldots i-1 \ldots) &= (\ldots i+1 \ldots i-1 \ldots i \ldots), \end{aligned} \tag{3.8}$$

and by acting as the identity if i occurs between $i-1$ and $i+1$. For instance, $\tilde{d}_3 \circ \tilde{d}_2(4123) = \tilde{d}_3(4123) = 3142$.

To decide when to apply d_i and when to use $\tilde{d}_i$, we appeal to the shifted content. Numbering the cells of a fixed $\boldsymbol{\nu}$ from 1 to n in shifted content reading order, let $\tilde{c}_i$ be the shifted content of the i^{th} cell. Define the weakly increasing word $\tau = \tau_1\tau_2\ldots\tau_n$ by

$$\tau_i = \max\{j \in [n] : \tilde{c}_j - \tilde{c}_i \leq k\}. \tag{3.9}$$

See Figure 5 for an example. To emphasize the relationship between τ and $\boldsymbol{\nu}$, we will sometimes write $\tau = \tau(\boldsymbol{\nu})$. Notice that there are finitely many possible τ of any fixed length n. Specifically, τ will always satisfy $\tau_n = n$ and $i \leq \tau_i \leq \tau_{i+1}$ for all $i < n$. Next, let $m(i)$ be the index of the the value in $\{i-1, i, i+1\}$ that occurs first in w, and let $M(i)$ be the index of the the value in $\{i-1, i, i+1\}$ that occurs last in w. We now define the desired involution,

$$D_i^{(\tau)}(w) := \begin{cases} d_i(w) & \tau_{m(i)} < M(i) \\ \tilde{d}_i(w) & \tau_{m(i)} \geq M(i). \end{cases} \tag{3.10}$$

As an example, we may take $\tau = 456667899$ and $w = 534826179$, as in Figure 5. Then $D_3^{(\tau)}(w) = \tilde{d}_i(w) = 542836179$ and $D_5^{(\tau)}(w) = d_i(w) = 634825179$.

Fig. 5: On the left, the shifted contents of a pair of skew diagrams with $\tau = 456667899$. On the right, a standard filling of the same tuple with shifted content word 534826179.

Direct inspection shows that if $\tau = \tau(\boldsymbol{\nu})$, then $D_i^{(\tau)}$ takes shifted content words of standard fillings of $\boldsymbol{\nu}$ to shifted content words of other standard fillings of $\boldsymbol{\nu}$. Thus, $D_i^{(\tau)}$ has a well defined action on SYT($\boldsymbol{\nu}$) inherited from the action of $D_i^{(\tau)}$ on shifted content words. We may then define the following.

Definition 3.3 Given some tuple of skew shapes $\boldsymbol{\nu}$, the *LLT graph* $\mathcal{L}_{\boldsymbol{\nu}} = (V, \sigma, E)$ is defined to be the (n, n)-signed colored graph with the following data:

1. $V = \{w \in S_n : w$ is the shifted content word of some $\mathbf{T} \in$ SYT($\boldsymbol{\nu}$)$\}$,
2. The signature function σ is given by the inverse descents of $w \in V$,
3. The edge sets E_i are defined by the nontrivial orbits of $D_i^{(\tau)}$ for all $1 < i < |\boldsymbol{\nu}|$, where $\tau = \tau(\boldsymbol{\nu})$.

Example 3.4 Consider $\boldsymbol{\nu} = ((2), (2), (1), (1))$. A portion of the LLT graph $\mathcal{L}_{\boldsymbol{\nu}}$ is presented in Figure 6. Here, $\mathcal{L}_{\boldsymbol{\nu}}$ is a subgraph of $\mathcal{G}_6^{(\tau)}$ with $\tau = 566666$. In the figure, the edge $\{312654, 412653\}$ is defined by the action of d_3 and d_4, while all other edges are defined by the action of $\tilde{d}_i$ for $1 < i < 6$.

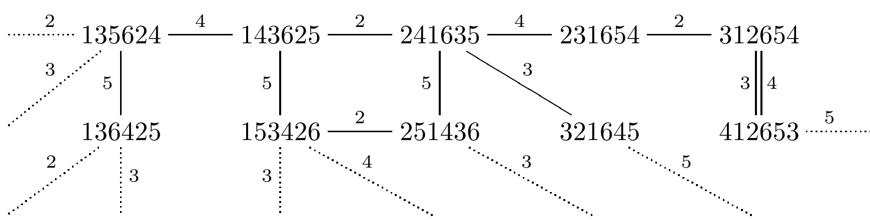

Fig. 6: A portion of $\mathcal{L}_{\boldsymbol{\nu}}$ with signatures omitted. Here $\boldsymbol{\nu} = ((2), (2), (1), (1))$.

While LLT graphs do not necessarily satisfy Axiom 4 or Axiom 6, they do satisfy a subset of the dual equivalence axioms. This is made precise in the following proposition.

Proposition 3.5 (Assaf (2011) Prop. 4.6) *Any LLT graph $\mathcal{L}_{\boldsymbol{\nu}}$ obeys Axioms 1, 2, 3, and 5. Furthermore, the* inv *statistic is constant on each connected component of $\mathcal{L}_{\boldsymbol{\nu}}$.*

To state the main theorem of this section, we will also need the following definition.

Definition 3.6 Given a k-tuple of skew shapes $\boldsymbol{\nu}$, let $S(\boldsymbol{\nu})$ be the set of distinct shifted contents of the cells in $\boldsymbol{\nu}$. Define the *diameter* of $\boldsymbol{\nu}$, denoted diam($\boldsymbol{\nu}$), as

$$\text{diam}(\boldsymbol{\nu}) := \max\{|R| : R \subset S(\boldsymbol{\nu}) \text{ and } |x - y| \leq k \text{ for all } x, y \in R\}.$$

See Figure 7 for an example.

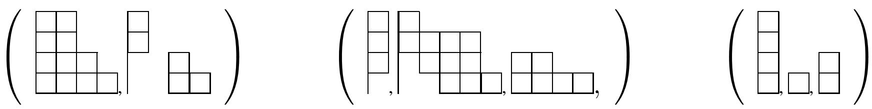

Fig. 7: The first two tuples of skew shapes have diameter 3. The third tuple of skew shapes has diameter 4.

Theorem 3.7 *The LLT graph $\mathcal{L}_{\boldsymbol{\nu}}$ is a dual equivalence graph if and only if* $\mathrm{diam}(\boldsymbol{\nu}) \leq 3$.

The proof of this theorem is primarily an application of Proposition 3.5, followed by a computer verification using Corollary 2.11. This computer verification can be found at

<http://www.math.washington.edu/∼austinis/Proof_LLTandDEG.sws>.

Finally, we give a lemma that allows us to find a representative vertex in each component of an LLT graph. It is crucial to results in the following section.

Lemma 3.8 *Let $\boldsymbol{\nu}$ be a tuple of skew shapes such that $|\boldsymbol{\nu}| = n$ and* $\mathrm{diam}(\boldsymbol{\nu}) \leq 3$. *Let $\mathcal{C}$ be any connected component of $\mathcal{L}_{\boldsymbol{\nu}}$, and let $\phi : \mathcal{C} \to \mathcal{G}_\lambda$ be an isomorphism, where $\lambda \vdash n$. Then there is exactly one standardized Yamanouchi word w in $\mathcal{C}$, and $P(w) = \phi(w) = U_\lambda$.*

3.3 The Schur Expansion when $\mathrm{diam}(\nu) \leq 3$

The goal of this section is to apply Theorem 3.7 to get specific Schur expansions for a family of LLT polynomials and a family of Macdonald polynomials.

Theorem 3.9 *Let $\boldsymbol{\nu}$ be any tuple of skew shapes with* $\mathrm{diam}(\boldsymbol{\nu}) \leq 3$. *Further, let $\mathbf{T}(\lambda)$ be the set of standard fillings in* $\mathrm{SYT}(\boldsymbol{\nu})$ *whose shifted content words are in* $\mathrm{SYam}(\lambda)$. *Then*

$$\widetilde{G}_{\boldsymbol{\nu}}(X;q) = \sum_{\lambda \vdash |\boldsymbol{\nu}|} \sum_{\mathbf{T} \in \mathbf{T}(\lambda)} q^{\mathrm{inv}(\mathbf{T})} s_\lambda.$$

In particular, the set of $\boldsymbol{\nu}$ such that $\mathrm{diam}(\boldsymbol{\nu}) \leq 3$ properly contains the set of $\boldsymbol{\nu}$ that are 2-tuples.

Proof Sketch: We begin by reducing Theorem 3.9 to a statement about signed colored graphs. Let $V_1, V_2, \ldots, V_m$ be the vertex sets of the connected components of $\mathcal{L}_{\boldsymbol{\nu}} = (V, \sigma, E)$. Then

$$\widetilde{G}_{\boldsymbol{\nu}}(X;q) = \sum_{\mathbf{T} \in \mathrm{SYT}(\boldsymbol{\nu})} q^{\mathrm{inv}_k(\mathbf{T})} F_{\sigma(\mathbf{T})}(X) = \sum_{v \in V} q^{\mathrm{inv}_{\boldsymbol{\nu}}(v)} F_{\sigma(v)}(X) = \sum_{j=1}^{m} \sum_{v \in V_j} q^{\mathrm{inv}_{\boldsymbol{\nu}}(v)} F_{\sigma(v)}(X), \quad (3.11)$$

We may then apply Theorem 3.7, (3.2), and Lemma 3.8 to express $\widetilde{G}_{\boldsymbol{\nu}}(X;q)$ in terms of standardized Yamanouchi words as given in the statement of the theorem. □

The next corollary follows immediately by applying Theorem 3.9 to the definition of modified Macdonald polynomials in (3.7). We also use the easily verified fact that tuples of ribbons in $\mathrm{TR}(\mu/\rho)$ have diameter less than or equal to three if and only if μ/ρ does not contain (3,3) or (4) as a subdiagram.

Corollary 3.10 *Let μ/ρ be a skew shape not containing $(3,3)$ or (4) as a subdiagram, and let $T(\lambda)$ be the set of bijective fillings of μ/ρ whose row reading words are in* $\mathrm{SYam}(\lambda)$. *Then*

$$\widetilde{H}_{\mu/\rho}(X;q,t) = \sum_{\lambda \vdash |\mu/\rho|} \sum_{T \in T(\lambda)} q^{\mathrm{inv}(T)} t^{\mathrm{maj}(T)} s_\lambda.$$

In particular, Corollary 3.10 applies to all $\tilde{H}_\mu(X;q,t)$ with $\mu_1 \leq 3$ where $\mu_2 \leq 2$.

Remark 3.11 The conditions on $\boldsymbol{\nu}$ and μ/ρ in Theorem 3.9 and Corollary 3.10, respectively, are sharp in the following sense. Let $\lambda = (2,2)$ and $\boldsymbol{\nu} = ((1),(1),(1,1))$ or $((1),(1),(1),(1))$. In particular, diam($\boldsymbol{\nu}$) $= 4$. Then

$$\widetilde{G}_{\boldsymbol{\nu}}(X;q)\big|_{q^4 s_\lambda} = 1 \quad \text{and} \quad \sum_{T \in T(\lambda)} q^{\mathrm{inv}(T)} s_\lambda\big|_{q^4 s_\lambda} = 2. \tag{3.12}$$

If $\lambda = (2,2)$ and $\mu/\rho = (4)$, then

$$\widetilde{H}_{\mu/\rho}(X;q,t)\big|_{q^4 s_\lambda} = 1 \quad \text{and} \quad \sum_{T \in T(\lambda)} q^{\mathrm{inv}(T)} t^{\mathrm{maj}(T)} s_\lambda\big|_{q^4 s_\lambda} = 2. \tag{3.13}$$

If $\lambda = (2,2,2)$ and $\mu/\rho = (3,3)$, then

$$\widetilde{H}_{\mu/\rho}(X;q,t)\big|_{q^3 t^3 s_\lambda} = 1 \quad \text{and} \quad \sum_{T \in T(\lambda)} q^{\mathrm{inv}(T)} t^{\mathrm{maj}(T)} s_\lambda\big|_{q^3 t^3 s_\lambda} = 2. \tag{3.14}$$

By using a symmetry of Macdonald polynomial, we also have the following immediate corollary.

Corollary 3.12 *Let μ/ρ be a skew shape not containing $(2,2,2)$ or $(1,1,1,1)$ as a subdiagram, and let $\widetilde{T}(\lambda)$ be the set of bijective fillings of $\tilde{\mu}/\tilde{\rho}$ whose row reading words are in* SYam(λ). *Then*

$$\widetilde{H}_{\mu/\rho}(X;q,t) = \sum_{\lambda \vdash |\mu/\rho|} \sum_{T \in \widetilde{T}(\lambda)} q^{\mathrm{maj}(T)} t^{\mathrm{inv}(T)} s_\lambda.$$

In particular, Corollary 3.10 applies to all $\tilde{H}_\mu(X;q,t)$ where μ has at most three rows and $\mu_3 \leq 1$.

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