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A Murnaghan-Nakayama Rule for Generalized Demazure Atoms

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Abstract. We prove an analogue of the Murnaghan-Nakayama rule to express the product of a power symmetric function and a generalized Demazure atom in terms of generalized Demazure atoms.

Résumé. Nous prouvons un analogue de la règle Murnaghan-Nakayama à exprimer le produit d'une fonction de puissance symétrique et un Demazure généralisée atomes en termes de généralisées atomes de Demazure.

Keywords: generalized Demazure atoms, nonsymmetric Macdonald polynomials, permuted basement fillings

1 Introduction

Haglund, Mason, and Remmel [HMR12] introduced a family of polynomials $\widehat{E}_{\gamma}^{\sigma}(x_1,\ldots,x_n)$ indexed by weak compositions γ of n and permutations σ in the symmetric group S_n that they called generalized Demazure atoms. The $\widehat{E}_{\gamma}^{\sigma}(x_1,\ldots,x_n)$ interpolate between the Schur functions and the Demazure atoms studied by Mason [Mas08]. The main goal of this paper is to develop a generalization of the Murnaghan-Nakayama rule to express the product of a power symmetric function $p_r(x_1,\ldots,x_n)$ and a generalized Demazure atom $\widehat{E}_{\gamma}^{\sigma}(x_1,\ldots,x_n)$ as a sum of generalized Demazure atoms $\widehat{E}_{\delta}^{\sigma}(x_1,\ldots,x_n)$. That is, we shall give a combinatorial definition of the coefficients $c_{\gamma,\sigma,\delta}^{(r)}$ where

$$p_r(x_1,\ldots,x_n)\widehat{E}_{\gamma}^{\sigma}(x_1,\ldots,x_n) = \sum_{\delta} c_{\gamma,\sigma,\delta}^{(r)}\widehat{E}_{\delta}^{\sigma}(x_1,\ldots,x_n). \tag{1}$$

Let $\mathbb{N}$ denote the set of natural numbers, $\{0,1,2,\ldots\}$, and let $\mathbb{P}$ denote the set of positive integers, $\{1,2,\ldots\}$. We say that $\lambda = (\lambda_1,\lambda_2,\ldots,\lambda_n)$ is a partition of m into n parts if each $\lambda_i \in \mathbb{N}$ with $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ and $\sum_{i=1}^n \lambda_i = m$. We say that $\gamma = (\gamma_1,\gamma_2,\ldots,\gamma_n)$ is a weak composition of m into n parts if each $\gamma_i \in \mathbb{N}$ and $\sum_{i=1}^n \gamma_i = m$. A composition of m is a weak composition which has no zeros. The *diagram* of γ, $dg(\gamma)$, is the set of m cells arranged in n columns so that there are γ_i cells in the i^{th} column and all the columns are flush with the bottom of the diagram. For example, the diagram of $\gamma = (2,0,1,0,3)$ is pictured in Figure 1. The *augmented diagram* of γ, $\widehat{dg}(\gamma)$, consists of $dg(\gamma)$ plus a row of n cells attached below. This lowest row is called the *basement*. Let $\lambda(\gamma)$ be the rearrangement of the parts of γ into weakly decreasing order. Thus, $\lambda(\gamma)$ produces the partition associated with each weak composition. For example, $\lambda(2,0,1,0,3) = (3,2,1,0,0)$.

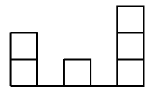

Fig. 1: The diagram of $\gamma = (2, 0, 1, 0, 3)$

Macdonald's well-known symmetric polynomials [Mac79], denoted $P_\lambda(x_1, x_2, \ldots, x_n; q, t)$ for λ a partition with n parts, have certain defining characteristics, including an orthogonality condition. Macdonald [Mac95] showed that many of these same characteristics were shared by a family of nonsymmetric polynomials indexed by weak compositions with n parts, denoted $E_\gamma(x_1, x_2, \ldots, x_n; q, t)$. These polynomials were given a combinatorial interpretation by Haglund, Haiman, and Loehr [HHL08] as the generating functions for fillings of the diagram of γ with positive integer entries satisfying some conditions.

Mason [Mas08], [Mas09] studied a slight variation of the $E_\gamma(x_1, x_2, \ldots, x_n; q, t)$ polynomials, called $\widehat{E}_\gamma(x_1, x_2, \ldots, x_n; q, t)$. The polynomial $\widehat{E}_\gamma$ is obtained from E_γ by reversing the order of the x_i's and sending q and t to their reciprocals. Mason considered the specialization arising from setting $q = t = 0$ in $\widehat{E}_\gamma(x_1, x_2, \ldots, x_n; q, t)$, hereafter referred to as $\widehat{E}_\gamma(x_1, x_2, \ldots, x_n)$. These polynomials arise in [LS90] as "standard bases" and are also called *Demazure atoms*. Using the work of [HHL08], Mason showed that $\widehat{E}_\gamma(x_1, x_2, \ldots, x_n)$ can be interpreted as the sum of the weights of certain fillings of $\hat{dg}(\gamma)$, which she called semi-standard augmented fillings. An important outcome of Mason's work is a generalization of the Robinson-Schensted-Knuth (RSK) insertion algorithm for semi-standard augmented fillings that she used to give combinatorial proofs of many results involving Demazure atoms. For example, this generalization of RSK was used to exhibit a bijection that showed that, for any partition β, the Schur function s_β could be expressed as

$$s_\beta(x_1, \ldots, x_n) = \sum_{\lambda(\gamma)=\beta} \widehat{E}_\gamma(x_1, \ldots, x_n). \tag{2}$$

Mason's work has led to several lines of further research. Using Mason's extension of RSK, Haglund, Luoto, Mason, and van Willigenburg [HLMvW11a] developed the theory for the quasisymmetric Schur functions, a new basis for the ring of quasisymmetric functions. In [HLMvW11b], they developed analogues of the Littlewood-Richardson rule to express the product of a Schur function and a Demazure atom (quasisymmetric Schur function) in terms of Demazure atoms (quasisymmetric Schur functions).

Haglund, Mason, and Remmel [HMR12] further generalized Mason's work by viewing semi-standard augmented fillings as fillings of augmented diagrams with entries in the basement equal to the identity permutation, ϵ_n. They also viewed reverse row-strict tableaux as fillings of augmented diagrams with entries in the basement equal to the reverse of the identity permutation, $\overline{\epsilon}_n$. Their work further generalizes Mason's extension of RSK to apply to fillings of diagrams with arbitrary permutations in the basement cells, called *permuted basement fillings*, or PBFs. The permuted basement fillings with basement σ generate the polynomials called *generalized Demazure atoms* and denoted $\widehat{E}^\sigma_\gamma(x_1, x_2, \ldots, x_n)$. The $\widehat{E}^\sigma_\gamma$'s can be viewed as intermediates between the Schur functions and the Demazure atoms. In fact, Haglund, Mason, and Remmel [HMR12] showed that for any permutation σ,

$$s_\beta(x_1, \ldots, x_n) = \sum_{\lambda(\gamma)=\beta} \widehat{E}^\sigma_\gamma(x_1, \ldots, x_n). \tag{3}$$

They also showed that $\widehat{E}^\sigma_\gamma(x_1, x_2, \ldots, x_n) = 0$ unless $\gamma_i \geq \gamma_j$ whenever $i < j$ and $\sigma_i > \sigma_j$. Note that when $\sigma = \epsilon_n$, we can recover (2) from (3). Also, when $\sigma = \overline{\epsilon}_n$, there is only one nonzero term in (3), so $s_\beta = \widehat{E}_\beta^{\overline{\epsilon}_n}$. We shall see that in the special case where $\sigma = \overline{\epsilon}_n$, our generalized Murnaghan-Nakayama rule reduces to the classical Murnaghan-Nakayama rule.

One of the motivations for studying the $\widehat{E}_\gamma^\sigma(x_1,\ldots,x_n)$ is an unpublished result of Haiman and Haglund which can be described briefly as follows. Let $\widehat{E}_\gamma^\sigma(x_1,\ldots,x_n;q,t)$ denote the polynomial obtained by modifying $\widehat{E}_\gamma(x_1,\ldots,x_n;q,t)$ in the following way. Under the interpretation of [HHL08] which associates $\widehat{E}_\gamma(x_1,\ldots,x_n;q,t)$ with fillings of $\widehat{dg}(\gamma)$, replace the basement permutation ϵ_n with $\sigma = \sigma_1\sigma_2\cdots\sigma_n$, and change nothing else. Then if $i+1$ occurs to the left of i in the basement $\sigma_1\sigma_2\cdots\sigma_n$, we have

$$T_i\widehat{E}_\gamma^\sigma(x_1,\ldots,x_n;q,t) = t^A\widehat{E}_\gamma^{\sigma'}(x_1,\ldots,x_n;q,t). \tag{4}$$

Here A equals one if the height of the column of $\widehat{dg}(\gamma)$ above $i+1$ in the basement is greater than or equal to the height of the column above i in the basement, and equals zero otherwise. Also, σ' is the permutation obtained by interchanging i and $i+1$ in σ. The T_i are generators for the affine Hecke algebra which act on monomials in the $\{x_1,x_2,\ldots,x_n\}$ by

$$T_ix^\lambda = tx^{s_i(\lambda)} + (t-1)\frac{x^\lambda - x^{s_i(\lambda)}}{1-x^{\alpha_i}},$$

with $x^{\alpha_i} = x_i/x_{i+1}$. See [HHL08] for a more detailed description of the T_i and their relevance to nonsymmetric Macdonald polynomials. The $\widehat{E}_\gamma^\sigma(x_1,\ldots,x_n)$ can be obtained by setting $q=t=0$ in $\widehat{E}_\gamma^\sigma(x_1,\ldots,x_n;q,t)$, and hence are a natural generalization of the $\widehat{E}_\gamma(x_1,\ldots,x_n)$ to investigate. If we set $q=t=0$ in the Hecke operator T_i, it reduces to a divided difference operator similar to those appearing in the definition of Schubert polynomials. By (4), $\widehat{E}_\gamma^\sigma(x_1,\ldots,x_n)$ can be expressed, up to a power of t, as a series of the divided difference operators applied to $\widehat{E}_\gamma^{\overline{\epsilon}_n}(x_1,\ldots,x_n)$.

Haglund, Mason, and Remmel [HMR12] proved analogues of the Pieri rules for generalized Demazure atoms. Let $h_r(x_1,\ldots,x_n)$ denote the r^{th} homogeneous symmetric function and $e_r(x_1,\ldots,x_n)$ denote the r^{th} elementary symmetric function. Then Haglund, Mason, and Remmel gave combinatorial interpretations to the coefficients $a^{(r)}_{\gamma,\sigma,\delta}$ and $b^{(r)}_{\gamma,\sigma,\delta}$ where

$$h_r(x_1,\ldots,x_n)\widehat{E}_\gamma^\sigma(x_1,\ldots,x_n) = \sum_\delta a^{(r)}_{\gamma,\sigma,\delta}\widehat{E}_\delta^\sigma(x_1,\ldots,x_n). \tag{5}$$

and

$$e_r(x_1,\ldots,x_n)\widehat{E}_\gamma^\sigma(x_1,\ldots,x_n) = \sum_\delta b^{(r)}_{\gamma,\sigma,\delta}\widehat{E}_\delta^\sigma(x_1,\ldots,x_n). \tag{6}$$

We will indicate how we can derive our combinatorial interpretation of the coefficients $c^{(r)}_{\gamma,\sigma,\delta}$ from these two rules. The same technique can be used to derive an analogue of the Murnaghan-Nakayama rule to express the product of a power symmetric function p_r and a quasisymmetric Schur function in terms of quasisymmetric Schur functions, but we will not pursue this topic in this paper.

The outline of this paper is as follows. In the next section, we will define permuted basement fillings and the generalized RSK insertion algorithm as well as introduce some results from [HMR12]. In section 3, we will state and prove our refinement of the Murnaghan-Nakayama rule.

2 PBFs and the Insertion Procedure

In this section, we shall formally define permuted basement fillings (PBFs) and the generalization of the RSK insertion algorithm due to Haglund, Mason, and Remmel [HMR12]. Let $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_n)$ be a weak composition of m into n parts. We will denote the cell in column i and row j of $\widehat{dg}(\gamma)$ by (i,j). The basement is row zero, and row indices increase from bottom to top. The leftmost column is column 1, and column indices increase moving from left to right. This way, it is easy to think of $\widehat{dg}(\gamma)$ as $dg(\gamma)$ augmented by row zero.

A filling F of an augmented diagram is a function $F : \widehat{dg}(\gamma) \to \mathbb{P}$, which can be pictured as an assignment of positive integers to the cells of $\widehat{dg}(\gamma)$. We will use $F(i,j)$ to denote the integer assigned to cell (i,j) by the function F. We will be interested only in fillings where the basement entries are a permutation $\sigma = \sigma_1\sigma_2\ldots\sigma_n$ of $\{1,2,\ldots,n\}$ and the column entries weakly decrease reading from bottom to top.

A set of cells $(i,k),(j,k),(i,k-1)$ in $\widehat{dg}(\gamma)$ is a *type A triple* if $i<j$, $k>0$, and $\gamma_i \geq \gamma_j$. A type A triple is an *inversion triple* in F if $F(j,k) < F(i,k) \leq F(i,k-1)$ or $F(i,k) \leq F(i,k-1) < F(j,k)$. A set of cells $(j,k+1),(i,k),(j,k)$ in $\widehat{dg}(\gamma)$ is a *type B triple* if $i<j$, $k \geq 0$, and $\gamma_i < \gamma_j$. A type B triple is an *inversion triple* in F if $F(i,k) < F(j,k+1) \leq F(j,k)$ or $F(j,k+1) \leq F(j,k) < F(i,k)$.

A filling F is said to satisfy the *B-increasing condition* if, whenever $i<l$ and $\gamma_i < \gamma_l$, it is true that $F(i,j-1) < F(l,j)$ for all $j \geq 1$.

A PBF F^σ of shape γ and basement σ is a filling of $\widehat{dg}(\gamma)$ with positive integer entries such that

1. the basement is filled with $\sigma_1, \sigma_2, ..., \sigma_n$ from left to right,
2. column entries weakly decrease reading from bottom to top,
3. every triple of type A or B is an inversion triple in F^σ, and
4. the B-increasing condition is satisfied.

It was observed in [HMR12] that the B-increasing condition plus the fact that column entries are weakly decreasing automatically implies that all type B triples are inversion triples.

The weight of a PBF F^σ with shape γ and basement σ is defined to be

$$wt(F^\sigma) = \prod_{(i,j)\in dg(\gamma)} x_{F^\sigma(i,j)}. \tag{7}$$

Note that the basement cells do not contribute to the weight of the PBF. The nonsymmetric polynomials $\widehat{E}^\sigma_\gamma(x_1,x_2,\ldots,x_n)$ are then defined by

$$\widehat{E}^\sigma_\gamma(x_1,x_2,\ldots,x_n) = \sum_{F^\sigma} wt(F^\sigma) \tag{8}$$

where the sum is over PBFs F^σ of shape γ and basement σ.

We say that a shape γ is *σ-compatible* if $\gamma_i \geq \gamma_j$ whenever $i<j$ and $\sigma_i > \sigma_j$. In [HMR12], it is shown that $\widehat{E}^\sigma_\gamma(x_1,x_2,\ldots,x_n) = 0$ unless γ is σ-compatible. Thus, there are no PBFs of shape γ and basement σ if γ is not σ-compatible.

When σ is the identity permutation, ϵ_n, a PBF is a semi-standard augmented filling as defined by Mason [Mas08], [Mas09]. When σ is the reverse of the identity permutation, $\overline{\epsilon}_n$, a PBF is strictly decreasing from left to right and weakly decreasing from bottom to top. The only $\overline{\epsilon}_n$-compatible shapes are partition shapes λ, so that PBFs can be described as "reverse row-strict tableaux." It is not hard to see that these reverse row-strict tableaux are in one-to-one correspondence with column-strict tableaux, so that for λ a partition, $\widehat{E}_{\lambda}^{\overline{\epsilon}_n} = s_{\lambda}$.

The *reading order* of the cells of $\widehat{dg}(\gamma)$ is obtained by moving across the rows from left to right, beginning with the highest row. Formally, (a, b) comes before (c, d) in reading order if $b > d$ or $b = d$ and $a < c$. As mentioned above, Mason defined an insertion procedure $k \to F$ analogous to the RSK insertion procedure that inserts a positive integer k into a semi-standard augmented filling to produce another semi-standard augmented filling. Haglund, Mason, and Remmel [HMR12] generalized this procedure to PBFs with arbitrary basements. To define this insertion $k \to F^{\sigma}$, let $\overline{F^{\sigma}}$ be the filling that extends the basement permutation by first adding a j in each cell $(j, 0)$ with $n < j \leq k$ and then adding an extra cell filled with a 0 on top of each column. Let $(x_1, y_1), (x_2, y_2), \ldots$ be the cells of $\overline{F^{\sigma}}$ listed in reading order. To insert k into F^{σ}, go through the cells of $\overline{F^{\sigma}}$ in reading order looking for the first (x_i, y_i) such that $\overline{F^{\sigma}}(x_i, y_i) < k \leq \overline{F^{\sigma}}(x_i, y_i - 1)$. Replace $\overline{F^{\sigma}}(x_i, y_i)$ with k and insert the cell's previous value into the remaining cells in reading order, beginning with (x_{i+1}, y_{i+1}). Continue in this way until some 0 is replaced by a positive integer. Finally, remove any zeros from the tops of the columns. Notice that $k \to F^{\sigma}$ creates a new cell at the top of some column in F^{σ}.

A fundamental result of [HMR12], used to prove many properties about PBFs, is the fact that this insertion procedure $k \to F^{\sigma}$ is well-defined and produces a PBF. Also, in the case that $\sigma = \overline{\epsilon}_n$, this insertion algorithm reduces to a reverse row-strict version of the usual RSK algorithm.

Another important question addressed in [HMR12] is whether this insertion procedure can be reversed. To answer this question, the authors define the term *removable cell*. Let γ and δ be weak compositions with n parts such that $dg(\gamma) \subseteq dg(\delta)$. We use $dg(\delta/\gamma)$ to denote the cells of $dg(\delta)$ which are not in $dg(\gamma)$. Suppose $dg(\delta/\gamma)$ consists of a single cell $c = (x, y)$. Then c is a removable cell from δ if there is no cell to the right of c that is at the top of a column in $dg(\delta)$. That is, there is no j with $x < j \leq n$ and $\delta_j = y$. It is shown that if F^{σ} is a PBF of shape γ and basement σ, and $G^{\sigma} = k \to F^{\sigma}$ is a PBF of shape δ, then the cell c in $dg(\delta/\gamma)$ is a removable cell. This terminology is used because it means that the insertion procedure can be reversed starting with c. That is, begin with the entry, say a, in cell c and read through the cells of G^{σ} in reverse reading order starting with c until an entry, say b, is found which is greater than a and positioned below an number less than or equal to a. Now replace b with a and continue reversing the insertion procedure with b. The entry that emerges from the first cell in reading order is the k which was originally inserted into F^{σ} to produce G^{σ}. So long as cell c is removable, the insertion can be reversed in this way.

A lemma used in [HMR12] to prove the Pieri rules for the products $h_r \widehat{E}_{\gamma}^{\sigma}$ and $e_r \widehat{E}_{\gamma}^{\sigma}$ will also be useful for our version of the Murnaghan-Nakayama rule. It is reproduced below:

Lemma 1 *Suppose that F^{σ} is a PBF, $G^{\sigma} = a \to F^{\sigma}$, and $H^{\sigma} = b \to G^{\sigma}$. Suppose F^{σ} is of shape α, G^{σ} is of shape β, and H^{σ} is of shape γ. Suppose A is the cell in $dg(\beta/\alpha)$ and B is the cell in $dg(\gamma/\beta)$. Then*

a. if $b \leq a$, then B is strictly above A, and

b. if $b > a$, then B is after A in reading order.

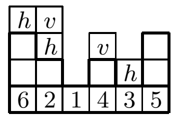

Fig. 2: A satisfactory labeling of $dg((3,3,0,2,1,2)/(2,1,0,1,0,2))$

3 Murnaghan-Nakayama Rule

The power symmetric function $p_r(x_1, x_2, \ldots, x_n)$ is defined as $\sum_{i=1}^{n} x_i^r$. Our goal is to express the product $p_r(x_1, x_2, \ldots, x_n)\widehat{E}_\gamma^\sigma(x_1, x_2, \ldots, x_n)$ as a sum of $\widehat{E}_\delta^\sigma$'s. In particular, we would like to find the coefficients $c_{\gamma,\sigma,\delta}^{(r)}$ in the expansion

$$p_r(x_1, x_2, \ldots, x_n)\widehat{E}_\gamma^\sigma(x_1, x_2, \ldots, x_n) = \sum_\delta c_{\gamma,\sigma,\delta}^{(r)} \widehat{E}_\delta^\sigma(x_1, x_2, \ldots, x_n). \tag{9}$$

Let γ be a weak composition of m with n parts, and let $\sigma \in S_n$. Let $\delta = (\delta_1, \ldots, \delta_n)$ be a weak composition of $m+r$ such that $dg(\gamma) \subseteq dg(\delta)$. Define a *satisfactory labeling of* $dg(\delta/\gamma)$ to be a labeling of the cells in $dg(\delta/\gamma)$ with v's and h's that is consistent with the following four rules:

(a) Assign an h to all but the rightmost cell in each row of $dg(\delta/\gamma)$.

(b) Assign a v to any cell above another cell of $dg(\delta/\gamma)$.

(c) Assign an h to any cell above a cell of $dg(\gamma)$ and having a cell of $dg(\delta/\gamma)$ one row below and anywhere to the left.

(d) Assign an h to the cells of the lowest row of $dg(\delta/\gamma)$.

For an example of a satisfactory labeling, see Figure 2

The following lemma gives a characterization for when $dg(\delta/\gamma)$ has a satisfactory labeling.

Lemma 2 *Let γ and δ be weak compositions such that $dg(\gamma) \subseteq dg(\delta)$. Then $dg(\delta/\gamma)$ has a satisfactory labeling if and only if there is no cell (x, y) in $dg(\delta/\gamma)$ such that $(x + j, y)$ and $(x, y - 1)$ are both cells in $dg(\delta/\gamma)$ for some $j > 0$. In other words, $dg(\delta/\gamma)$ has a satisfactory labeling if and only if it avoids the configuration.*

A satisfactory labeling of $dg(\delta/\gamma)$ is called a *satisfactory k-hook labeling* if it produces $k + 1$ h's and $r - (k + 1)$ v's and meets the following two additional conditions. Let $c_1 = (x_1, y_1), \ldots, c_{k+1} = (x_{k+1}, y_{k+1})$ be the cells labeled h listed in reading order. Let $c_{k+2} = (x_{k+2}, y_{k+2}), \ldots, c_r = (x_r, y_r)$ be the cells labeled v listed in reverse reading order. Let $dg(\delta^{(i)})$ be the diagram of γ plus the cells $c_1, c_2, \ldots, c_i$. A satisfactory k-hook labeling must also satisfy, for $i = 1, 2, \ldots, r$,

1. $\delta^{(i)}$ is a σ-compatible weak composition shape and
2. c_i is a removable cell from $\delta^{(i)}$.

Fig. 3: Unique satisfactory labelings of $dg((5,0,1)/(2,0,1))$ and $dg((3,0,3)/(2,0,1))$

Note that rule (d) in the definition of a satisfactory labeling requires that there is at least one h, so a satisfactory k-hook labeling has $k \geq 0$.

We say that the shape δ/γ is a γ*-transposed* k*-hook relative to basement* σ if rules (a)-(d) can be used to assign v and h labels to all of the cells of $dg(\delta/\gamma)$ in a unique way and that labeling is a satisfactory k-hook labeling. With these definitions, we can state the main theorem.

Theorem 1 *If* $\gamma = (\gamma_1, \ldots, \gamma_n)$ *is a weak composition of* m *and* $\sigma \in S_n$*, then*

$$p_r(x_1, x_2, \ldots, x_n)\widehat{E}_\gamma^\sigma(x_1, x_2, \ldots, x_n) = \sum_\delta (-1)^k \widehat{E}_\delta^\sigma(x_1, x_2, \ldots, x_n) \tag{10}$$

where the sum is over all weak compositions $\delta = (\delta_1, \ldots, \delta_n)$ *of* $m+r$ *such that* $dg(\gamma) \subseteq dg(\delta)$ *and* δ/γ *is a* γ*-transposed* k*-hook relative to basement* σ.

Theorem 1 says that the nonzero coefficients $c^{(r)}_{\gamma,\sigma,\delta}$ in (9) are ± 1, depending on the number of h's in the unique satisfactory k-hook labeling of $dg(\delta/\gamma)$.

For example, consider the product $p_3(x_1, x_2, x_3)\widehat{E}^{312}_{(2,0,1)}(x_1, x_2, x_3)$. The shapes δ that appear on the right hand side of (10) must be 312-compatible compositions of 6 into 3 parts containing $(2, 0, 1)$. Since the first and second basement entries are out of order, we require $\delta_1 \geq \delta_2$. Similarly, we require $\delta_1 \geq \delta_3$. Take $dg(2, 0, 1)$ and consider all the ways to add three cells around the outside of the diagram. Adding three cells to the first column creates the 312-compatible shape $(5, 0, 1)$. Moreover, this shape has a unique satisfactory labeling as shown in Figure 3, and it is easy to check that this is a satisfactory 0-hook labeling. Thus, $\widehat{E}^{312}_{(5,0,1)}(x_1, x_2, x_3)$ appears on the right hand side of (10) with coefficient $(-1)^0 = 1$. If instead we add two cells to the first column and one cell to another column, the lower of the two cells in the first column will not be assigned a v or h label by rules (a)-(d). Thus, there will be more than one satisfactory labeling of the diagram of such a shape, and it will not contribute to the sum. If one cell is added to the first column and two cells are added elsewhere, note that rules (b) through (d) will not apply to the highest cell in the first column. Therefore, in order to get a unique satisfactory labeling, one where each cell's label is forced by rules (a) through (d), we must make sure that the highest cell in the first column is not the rightmost cell in its row. The only shape that achieves this requirement is $(3, 0, 3)$. This shape has a unique satisfactory labeling as shown in Figure 3, and this labeling is a satisfactory 1-hook labeling. So $\widehat{E}^{312}_{(3,0,3)}(x_1, x_2, x_3)$ appears on the right hand side of (10) with coefficient $(-1)^1 = -1$. Finally, if we add no cells to the first column, the only way to keep the height of the first column greater than or equal to the heights of the other columns, as required by the basement permutation 312, is to create the shape $(2, 2, 2)$. In this case, however, $dg((2, 2, 2)/(2, 0, 1))$ contains the configuration. Lemma 2 says there is no satisfactory labeling, so this shape will not contribute to the sum. Putting this all together gives

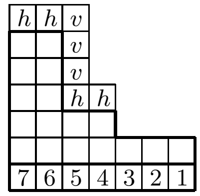

Fig. 4: A rim hook and its satisfactory labeling

$p_3(x_1,x_2,x_3)\widehat{E}^{312}_{(2,0,1)}(x_1,x_2,x_3) = \widehat{E}^{312}_{(5,0,1)}(x_1,x_2,x_3) - \widehat{E}^{312}_{(3,0,3)}(x_1,x_2,x_3)$, which can be verified by direct computation.

Note that in the special case where $\sigma = \overline{\epsilon}_n$ and γ is actually a partition shape, $\widehat{E}^{\sigma}_{\gamma} = s_\gamma$ by (3). Our rule then reduces to the reverse row-strict version of the classical Murnaghan-Nakayama rule, which says

$$p_r(x_1,x_2,\ldots,x_n)s_\gamma(x_1,x_2,\ldots,x_n) = \sum_{\lambda}(-1)^{w(\lambda/\gamma)-1}s_\lambda(x_1,x_2,\ldots,x_n) \tag{11}$$

Here, the sum is over all partition shapes λ such that $dg(\lambda/\gamma)$ is a *rim hook* of size r, that is a connected skew shape with no 2×2 square. Also, the width of λ/γ, $w(\lambda/\gamma)$, is the number of columns over which the cells of $dg(\lambda/\gamma)$ stretch.

First, if λ is a shape that appears on the right hand side of (11), then it also appears on the right hand side of (10) with the same coefficient. If $dg(\lambda/\gamma)$ is a connected skew shape around the outside of $dg(\gamma)$ avoiding the 2×2 square, then every cell of $dg(\lambda/\gamma)$ either has another cell of $dg(\lambda/\gamma)$ to its right or below it, with the exception of the last cell in reading order. This means that rule (d) labels the last cell in reading order, and rules (a) and (b) label all the other cells of $dg(\lambda/\gamma)$ in a unique way, as in Figure 4. Let $c_1 = (x_1,y_1),\ldots,c_{k+1} = (x_{k+1},y_{k+1})$ be the cells labeled h listed in reading order. Let $c_{k+2} = (x_{k+2},y_{k+2}),\ldots,c_r = (x_r,y_r)$ be the cells labeled v listed in reverse reading order. Let $dg(\lambda^{(i)})$ be the diagram of γ plus the cells $c_1,c_2,\ldots,c_i$. Clearly, each $\lambda^{(i)}$ is a partition shape, making it compatible with basement $\overline{\epsilon}_n$. Also, each c_i is on the rightmost column of its height in $\lambda^{(i)}$, making each c_i a removable cell. This means the unique labeling is a satisfactory k-hook labeling, so λ/γ is a γ-transposed k-hook relative to basement $\overline{\epsilon}_n$. Then this s_λ appears as a term in the sum with coefficient $(-1)^k$. This coefficient is the same as that given in the classical Murnaghan-Nakayama rule, because the width of λ/γ is the number of cells labeled h. The labeling assigns an h to $k+1$ cells so that $(-1)^{w(\lambda/\gamma)-1} = (-1)^k$.

Now, if λ is not counted by (11), it is also not counted by (10). If $dg(\lambda/\gamma)$ is not a connected skew shape, then consider the first connected component of $dg(\lambda/\gamma)$ in reading order. The last cell in reading order of this component will not have its label forced by rules (a) through (d). It is neither above nor to the left of another cell. It is not in the lowest row of $dg(\lambda/\gamma)$ because there is another component following this one, and the situation described by rule (c) does not apply to partition shapes. Thus, there is not a unique satisfactory of labeling of $dg(\lambda/\gamma)$, meaning δ/γ is not a γ-transposed k-hook relative to basement $\overline{\epsilon}_n$, so s_λ will not appear as a term in the sum. Similarly, if $dg(\lambda/\gamma)$ contains a 2×2 square, Lemma 2 says that $dg(\lambda/\gamma)$ has no satisfactory labeling. In this case, clearly λ/γ is not a γ-transposed k-hook relative to basement $\overline{\epsilon}_n$. Thus the shapes λ for which λ/γ is a γ-transposed k-hook relative to basement $\overline{\epsilon}_n$ are exactly the shapes λ such that $dg(\lambda/\gamma)$ is a rim hook.

To prove Theorem 1, we start with a well-known result from symmetric function theory, which says that the power symmetric functions can be expressed as an alternating sum of hook Schur functions:

$$p_r(x_1, x_2, \ldots, x_n) = \sum_{k=0}^{r-1} (-1)^k s_{(r-k,1^k)}. \tag{12}$$

Using (12) allows us to write

$$p_r(x_1, x_2, \ldots, x_n) \widehat{E}_\gamma^\sigma(x_1, x_2, \ldots, x_n) = \sum_{k=0}^{r-1} (-1)^k s_{(r-k,1^k)} \widehat{E}_\gamma^\sigma(x_1, x_2, \ldots, x_n) \tag{13}$$

thereby reducing the problem to one of multiplying a hook Schur function and a generalized Demazure atom.

If $A_k(\delta/\gamma)$ is the number of satisfactory k-hook labelings of $dg(\delta/\gamma)$, the following lemma answers the question of how to multiply a hook Schur function and a generalized Demazure atom. While it is possible to give a proof of this lemma using the Pieri rules of [HMR12], we have chosen to give a direct proof here.

Lemma 3 *If $\gamma = (\gamma_1, \ldots, \gamma_n)$ is a weak composition of m and $\sigma \in S_n$, then*

$$s_{(r-k,1^k)}(x_1, x_2, \ldots, x_n) \widehat{E}_\gamma^\sigma(x_1, x_2, \ldots, x_n) = \sum_\delta A_k(\delta/\gamma) \widehat{E}_\delta^\sigma(x_1, x_2, \ldots, x_n) \tag{14}$$

where the sum is over all weak compositions $\delta = (\delta_1, \ldots, \delta_n)$ of $m + r$ such that $dg(\gamma) \subseteq dg(\delta)$ and δ/γ has a satisfactory k-hook labeling.

Proof: The left hand side of (14) can be interpreted as the sum of the weights of all pairs (w, F^σ), where F^σ is a PBF of shape γ with basement σ and $w = w_1 w_2 \ldots w_r$ satisfies $1 \leq w_1 < \cdots < w_k < w_{k+1} \leq n$ and $n \geq w_{k+1} \geq \cdots \geq w_r \geq 1$. This condition on w comes from interpreting $s_{(r-k,1^k)}$ as the sum of the weights of all reverse row-strict tableaux of shape $(r-k, 1^k)$ read by columns. Here, the weight of a pair (w, F^σ) is defined to be $wt(F^\sigma) \prod_{i=1}^r x_{w_i}$. The right hand side of (14) can be interpreted as the sum of the weights of all pairs (L, G^σ) where G^σ is a PBF with basement σ and shape $\delta = (\delta_1, \ldots, \delta_n)$ of size $m + r$ such that $dg(\gamma) \subseteq dg(\delta)$ and L is a satisfactory k-hook labeling of $dg(\delta/\gamma)$. Here, the weight of a pair (L, G^σ) is just $wt(G^\sigma)$.

Let Θ be the function which takes such a (w, F^σ) pair to the pair (L, G^σ) where G^σ is the PBF that results from the insertion of w into F^σ and L is the labeling induced by this insertion, defined below. Let δ be the shape of this resulting PBF G^σ. Let $G_0^\sigma = F^\sigma$ and $G_i^\sigma = w_1 \ldots w_i \to F^\sigma$ for $i = 1, \ldots, r$. Let $\delta^{(i)}$ be the shape of G_i^σ, and let c_i be the cell in $dg(\delta^{(i)}/\delta^{(i-1)})$. To obtain the induced labeling L of $dg(\delta/\gamma)$, simply label cells $c_1, \ldots, c_{k+1}$ with h, and label cells $c_{k+2}, \ldots, c_r$ with v. Note that by Lemma 1, $c_1, \ldots, c_{k+1}$ appear in reading order because $w_1 < \cdots < w_k < w_{k+1}$, and $c_{k+2}, \ldots, c_r$ appear in reverse reading order because $w_{k+2} \geq \cdots \geq w_r$. Also, L produces the correct number of h's and v's to be a satisfactory k-hook labeling. We must check that the two additional conditions are met, as well as that L is a satisfactory labeling.

We know that for each $i = 1, \ldots, r$, $\delta^{(i)}$ is a σ-compatible weak composition of $m + i$ because it arises from insertion. Also, insertion creates only removable cells, so each c_i is a removable cell. This means conditions 1 and 2 in the definition of a satisfactory k-hook labeling hold. It remains to show that the labeling L satisfies the definition of a satisfactory labeling.

First, note that none of $\{c_{k+2}, \ldots, c_r\}$ have another cell in $dg(\delta/\gamma)$ to their right in the same row. Suppose that some c_l with $k+2 \leq l \leq r$ has another cell c_i to its right. Since c_l is removable, it must be the case that $i > l$. This means that $w_i \leq w_l$ so by Lemma 1, c_i should be strictly above c_l. Since the assumption is that c_i and c_l are in the same row, it must not be possible for any of $\{c_{k+2}, \ldots, c_r\}$ to have another cell in $dg(\delta/\gamma)$ to the right. This means every cell with another cell to the right must be among $\{c_1, \ldots, c_{k+1}\}$, and our labeling L assigns them all h. Thus, part (a) of the definition holds.

Next, note that none of $\{c_1, \ldots, c_{k+1}\}$ have another cell in $dg(\delta/\gamma)$ immediately below. Suppose that some c_l with $1 \leq l \leq k+1$ is directly above some other c_i. For $\delta^{(l)}$ to be a weak composition shape, it must be the case that $i \leq l$. This means $w_i < w_l$ so that c_l should come after c_i in reading order by Lemma 1. This contradicts the fact that c_l is above c_i, which means all cells with another cell directly below must be among $\{c_{k+2}, \ldots, c_r\}$ and our labeling assigns them all v, as required by part (b) of the definition.

Now suppose that some c_l with $k+2 \leq l \leq r$ is above a cell of $dg(\gamma)$ and has some other $c_i \in dg(\delta/\gamma)$ one row below and to the left. Then $i > l$ otherwise c_i would not be removable. Then $w_l \geq w_i$ so c_i should be strictly above c_l. Since this is not the case, it must not be possible for one of $\{c_{k+2}, \ldots, c_r\}$ to be above a cell in $dg(\gamma)$ and have some other cell $dg(\delta/\gamma)$ one row below and to the left. Any such cell must therefore be among $\{c_1, \ldots, c_{k+1}\}$, and L assigns them all h. Thus, part (c) of the definition holds.

Finally, we show that none of $\{c_{k+2}, \ldots, c_r\}$ can end up in the lowest row of $dg(\delta/\gamma)$. For any w_l with $k+2 \leq l \leq r$, $w_l \leq w_{k+1}$ so c_l must fall strictly above c_{k+1} in $dg(\delta/\gamma)$. Since each such c_l must fall strictly above another cell in the diagram, none can be in the bottom row. Therefore, all cells in the bottom row of the diagram are among $\{c_1, \ldots, c_{k+1}\}$. The labeling L assigns them all h, thereby satisfying part (d) of the definition.

We have shown that G^σ is a PBF with basement σ and shape $\delta = (\delta_1, \ldots, \delta_n)$ of size $m+r$ such that $dg(\gamma) \subseteq dg(\delta)$ and L is a satisfactory k-hook labeling. Also, the weight of the pair (w, F^σ) is clearly the same as the weight of G^σ. Since the insertion procedure can be reversed, the map Θ is one-to-one.

It remains to show that Θ is surjective, or that for each G^σ with a given satisfactory k-hook labeling, there is a pair (w, F^σ) that maps to it under Θ. Suppose G^σ is a PBF with basement σ and shape $\delta = (\delta_1, \ldots, \delta_n)$ of size $m+r$ such that $dg(\gamma) \subseteq dg(\delta)$ and L is a satisfactory k-hook labeling of $dg(\delta/\gamma)$. Label the cells with v's and h's according to L. Then label the h's in reading order with $c_1, \ldots, c_{k+1}$ and the v's in reverse reading order with $c_{k+2}, \ldots, c_r$. Since c_r is a removable cell, we can reverse the insertion procedure to produce a PBF which we will call F^σ_{r-1} and a letter w_r such that $G^\sigma = w_r \to F^\sigma_{r-1}$. Since each c_i is removable, we can continue to reverse the insertion procedure starting with c_{r-1} next, all the way down to c_1. Each step in this reversal produces a new PBF F^σ_i and a letter w_{i+1} such that $G^\sigma = w_{i+1} \ldots w_r \to F^\sigma_i$. The shape of F^σ_i is δ with the cells $c_{i+1}, \ldots, c_r$ removed. This means F^σ_0 is a PBF of shape γ and basement σ such that $w = w_1 \ldots w_r$ inserted into F^σ_0 produces G^σ and induces the labeling L.

We must show, however, that the resulting word $w = w_1 \ldots w_r$ satisfies $1 \leq w_1 < \cdots < w_k < w_{k+1} \leq n$ and $n \geq w_{k+1} \geq \cdots \geq w_r \geq 1$. It is clear that each w_i satisfies $1 \leq w_i \leq n$, as w_i was an element of a PBF with basement $\sigma \in S_n$. Suppose for a contradiction that some $w_i \geq w_{i+1}$ for $1 \leq i \leq k$. Then Lemma 1 says that c_{i+1} should be strictly above c_i, which contradicts the fact that $\{c_1, \ldots, c_{k+1}\}$ were labeled in reading order. So $1 \leq w_1 < \cdots < w_k < w_{k+1} \leq n$. Now suppose that some $w_i < w_{i+1}$ for $k+1 \leq i \leq r$. Then Lemma 1 says that c_{i+1} should appear after c_i in reading order, which contradicts the fact that $\{c_{k+1}, \ldots, c_r\}$ were labeled in reverse reading order. Thus, it is also true that $n \geq w_{k+1} \geq \cdots \geq w_r \geq 1$. Therefore, Θ is a bijection, which proves (14).

$\square$

Replacing 14 in 13 allows us to write

$$p_r(x_1, x_2, \ldots, x_n)\widehat{E}^{\sigma}_{\gamma}(x_1, x_2, \ldots, x_n) = \sum_{k=0}^{r-1}(-1)^k \sum_{\delta} A_k(\delta/\gamma)\widehat{E}^{\sigma}_{\delta}(x_1, x_2, \ldots, x_n) \tag{15}$$

where the sum is over all weak compositions $\delta = (\delta_1, \ldots, \delta_n)$ of $m + r$ such that $dg(\gamma) \subseteq dg(\delta)$ and $dg(\delta/\gamma)$ has a satisfactory k-hook labeling. Next we state a technical lemma for which we will not supply a proof due to lack of space. This lemma will help us define an involution which will allow us to simplify the right hand side of (15).

Lemma 4 *Suppose δ is a shape such that $dg(\delta/\gamma)$ has a satisfactory j-hook labeling for some j and rules (a) through (d) cannot be used to assign labels to all cells of $dg(\delta/\gamma)$. Then every satisfactory labeling of $dg(\delta/\gamma)$ is a satisfactory k-hook labeling for some k.*

Using these lemmas, we can prove Theorem 1.

Proof of Theorem 1: We have from (15) an expression for $p_r\widehat{E}^{\sigma}_{\gamma}$ in terms of $\widehat{E}^{\sigma}_{\delta}$'s. Suppose δ is a shape such that $\widehat{E}^{\sigma}_{\delta}$ appears on the right hand side of (15). By Lemma 3, δ is a shape such that $dg(\delta/\gamma)$ has a satisfactory j-hook labeling for some j. Suppose also that δ/γ is not a γ-transposed k-hook relative to basement σ, so that there is more than one way to label its cells in accordance with rules (a) through (d).

We will associate with each satisfactory labeling L of $dg(\delta/\gamma)$ the sign $(-1)^k$ if L is a satisfactory k-hook labeling. Note that Lemma 4 implies that each satisfactory labeling L is a satisfactory k-hook labeling for some value of k, so this notion of sign is well-defined. Now define an involution I on the set of satisfactory labelings of $dg(\delta/\gamma)$. Take any satisfactory labeling L of $dg(\delta/\gamma)$. By Lemma 4, L is a satisfactory k-hook labeling for some k. To define $I(L)$, take the first cell c in reading order of $dg(\delta/\gamma)$ which was not forced to be a v or an h by rules (a) through (d). If L labeled cell c with an h, change it to a v. Otherwise, change it from a v to an h. This new labeling is $I(L)$. $I(L)$ is still a satisfactory labeling because we have not changed the label of any cell to which rules (a) through (d) apply. The number of h's in $I(L)$ is either one fewer or one more than the number of h's in L. By Lemma 4, $I(L)$ is a satisfactory $k+1$-hook or $k-1$-hook labeling. Furthermore the sign of $I(L)$ is the opposite of the sign of L. I is an involution because I applied to $I(L)$ will change the label of the same cell c.

Applying this involution to the labelings of $dg(\delta/\gamma)$ when δ/γ is not a γ-transposed k-hook relative to basement σ gives a way to pair $\widehat{E}^{\sigma}_{\delta}$ terms with opposite signs on the right hand side of (15). Since I has no fixed points, all such $\widehat{E}^{\sigma}_{\delta}$ terms will cancel. This means that if δ is a shape for which any cells of $dg(\delta/\gamma)$ are left undetermined by rules (a)-(d), then $\widehat{E}^{\sigma}_{\delta}$ will not appear in the expansion of $p_n\widehat{E}^{\sigma}_{\gamma}$. The terms that do appear are for those δ's for which rules (a)-(d) assign a v or h label to every cell of $dg(\delta/\gamma)$, which are exactly γ-transposed k-hooks relative to basement σ. $\square$

If $\lambda = (\lambda_1, \ldots, \lambda_s)$ is a partition, this theorem can be used to multiply $p_\lambda\widehat{E}^{\sigma}_{\gamma}$ by first writing $p_\lambda = p_{\lambda_1}p_{\lambda_2}\ldots p_{\lambda_s}$ and then repeatedly applying 10.

In conclusion, we note that one can define a quasisymmetric Schur function QS_α and a row-strict quasisymmetric Schur function RS_α for each composition α of n; see [HLMvW11a] and [MR11], respectively. The methods of this paper can easily be modified to prove analogues of the Murnaghan-Nakayama rule for quasisymmetric Schur functions and row-strict quasisymmetric Schur functions. That is, we can give combinatorial interpretations to the coefficients $u_{\alpha,\beta}^{(r)}$ and $v_{\alpha,\beta}^{(r)}$ where

$$\begin{aligned} p_r(x_1,\ldots,x_n)QS_\alpha(x_1,\ldots,x_n) &= \sum_\beta u_{\alpha,\beta}^{(r)} QS_\beta(x_1,\ldots,x_n) \text{ and} \\ p_r(x_1,\ldots,x_n)RS_\alpha(x_1,\ldots,x_n) &= \sum_\beta v_{\alpha,\beta}^{(r)} RS_\beta(x_1,\ldots,x_n). \end{aligned}$$

This work will appear in a subsequent paper.

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