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Antipode and Convolution Powers of the Identity in Graded Connected Hopf Algebras

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Abstract. We study convolution powers id^{*n} of the identity of graded connected Hopf algebras H. (The antipode corresponds to $n = -1$.) The chief result is a complete description of the characteristic polynomial—both eigenvalues and multiplicity—for the action of the operator id^{*n} on each homogeneous component H_m. The multiplicities are independent of n. This follows from considering the action of the (higher) Eulerian idempotents on a certain Lie algebra $\mathfrak{g}$ associated to H. In case H is cofree, we give an alternative (explicit and combinatorial) description in terms of palindromic words in free generators of $\mathfrak{g}$. We obtain identities involving partitions and compositions by specializing H to some familiar combinatorial Hopf algebras.

Résumé. Nous étudions les puissances de convolution id^{*n} de l'identité d'une algèbre de Hopf graduée et connexe H quelconque. (L'antipode correspond à $n = -1$.) Le résultat principal est une description complète du polynôme caractéristique (des valeurs propres et de leurs multiplicités) de l'opérateur id^{*n} agissant sur chaque composante homogène H_m. Les multiplicités sont indépendants de n. Ceci résulte de l'examen de l'action des idempotents eulériens (supérieures) sur une algèbre de Lie $\mathfrak{g}$ associé à H. Dans le cas où H est colibre, nous donnons une description alternative (explicite et combinatoire) en termes de mots palindromes dans les générateurs libres de $\mathfrak{g}$. Nous obtenons des identités impliquant des partitions et compositions en choisissant comme H certaines algèbres de Hopf combinatoires connues.

Keywords: Hopf power, antipode, Eulerian idempotent, graded connected Hopf algebra, Schur indicator.

Dedicated to the memory of Jean-Louis Loday.

1 Introduction

As the practice of algebraic combinatorics often involves breaking and joining like combinatorial structures (planar trees, permutations, set partitions, etc.), it is right to say that bialgebras are ubiquitous in the theory. This was the argument put forth by G.C. Rota and others, and increasingly, researchers are taking it to heart. On the other hand, the defining property of "Hopf algebra"—the existence of the *antipode*—is

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less often explicitly considered. To be sure, there is a general result stating that the bialgebras built within algebraic combinatorics are *automatically* Hopf algebras (see Section 2).

The *antipode problem* (Aguiar and Mahajan, 2013, Section 5.4) asks for explicit knowledge of the antipode. This can be a source of interesting combinatorial results. Consider Lam et al. (2011), where the antipode plays a crucial role in proving a skew-Littlewood Richardson rule conjectured in Assaf and McNamara (2011). Here is a small illustration of the utility of the antipode (an application we obtain in Section 6.1). If $p_k(n)$ denotes the number of partitions of length k of a positive integer n, and $c(n)$ denotes the number of self-conjugate partitions of n, then

$$(-1)^n\, c(n) = \sum_{k=1}^{n} (-1)^k\, p_k(n).$$

If a Hopf algebra H is commutative or cocommutative, then it is well-known that its antipode $S\colon H \to H$ is an involution: $S^2 = \mathrm{id}$. In particular, its eigenvalues are ± 1. We prove in Corollary 5 that in case H is graded connected, the eigenvalues of the antipode are always ± 1, regardless of (co)commutativity, even if S may have infinite order on any homogeneous component. This is a consequence of our main result (Theorem 4), which provides a complete description of the characteristic polynomial for the convolution power id^{*n} acting on each homogeneous component of H. (The antipode satisfies $S = \mathrm{id}^{*(-1)}$.)

This note is organized as follows. In Section 2, we introduce the Hopf and Lie preliminaries needed to state and prove Theorem 4, which is carried out in Section 3. In Section 4, we give two refinements of our main result in the presence of additional (co)freeness assumptions. Section 5 applies the preceding to higher Schur indicators, and Section 6 provides illustrations of the results and derives some applications.

2 Hopf and Lie preliminaries

Throughout, we assume $\Bbbk$ is a field of characteristic zero. A Hopf algebra is a vector space H over $\Bbbk$ with a host of maps—product ($\mu\colon H \otimes H \to H$), unit ($\iota\colon \Bbbk \to H$), coproduct ($\Delta\colon H \to H \otimes H$), counit ($\varepsilon\colon H \to \Bbbk$), and antipode ($S\colon H \to H$)—satisfying various compatibility axioms, *e.g.*, Δ and ε are algebra maps. The convolution product of two linear maps $P, Q\colon H \to H$ is defined by $P * Q := \mu \circ (P \otimes Q) \circ \Delta$. This is an associative product, making $\mathrm{End}(H)$ into a $\Bbbk$-algebra, with unit element $\iota\varepsilon$. The antipode is the convolution-inverse of the identity map id; that is, $S * \mathrm{id} = \iota\varepsilon = \mathrm{id} * S$.

2.1 Coradical filtration and primitive elements

Let $H_{(0)}$ denote the *coradical* of a Hopf algebra H. This is the sum of the simple subcoalgebras of H. Given any two subspaces U, V of H, define their wedge by

$$U \wedge V := \Delta^{-1}(U \otimes H + H \otimes V). \tag{1}$$

Putting $H_{(n)} = H_{(0)} \wedge H_{(n-1)}$ for all $n \geq 1$ affords H with the *coradical filtration:*

$$H_{(0)} \subseteq H_{(1)} \subseteq \cdots \subseteq H_{(n)} \subseteq \cdots H \quad \text{and} \quad H = \bigcup_{n \geq 0} H_{(n)}. \tag{2}$$

The unit element (as well as any other *group-like* element) of H belongs to $H_{(0)}$. The Hopf algebra H is *connected* if $H_{(0)}$ is spanned by the unit element. In this case, $H_{(1)} = H_{(0)} \oplus \mathcal{P}(H)$, where

$$\mathcal{P}(H) = \{x \in H \mid \Delta(x) = 1 \otimes x + x \otimes 1\}$$

is the space of *primitive elements* of H. It is a Lie subalgebra of H under the commutator bracket.

If $\mathfrak{g}$ is a Lie algebra, its universal enveloping algebra $U(\mathfrak{g})$ is a Hopf algebra for which the space of primitive elements is $\mathfrak{g}$. If H is connected and cocommutative, the Cartier–Milnor–Moore (CMM) theorem states that H is isomorphic as Hopf algebra to $U(\mathcal{P}(H))$. The Poincaré–Birkhoff–Witt (PBW) identifies the vector space $U(\mathfrak{g})$ with $S(\mathfrak{g})$, the symmetric algebra on $\mathfrak{g}$.

We associate a commutative Hopf algebra to any connected Hopf algebra. This will enable us to use the above Lie machinery in a wider class of algebras. Given a Hopf algebra with coradical filtration $H = \bigcup_{n\geq 0} H_{(n)}$, let $\operatorname{gr} H$ denote the associated graded space

$$\operatorname{gr} H = H_{(0)} \oplus \big(H_{(1)}\big/H_{(0)}\big) \oplus \big(H_{(2)}\big/H_{(1)}\big) \oplus \big(H_{(3)}\big/H_{(2)}\big) \oplus \cdots . \tag{3}$$

It is a graded Hopf algebra for which the component of degree n is $H_{(n)}\big/H_{(n-1)}$ (Montgomery, 1993, Ch. 5). If H is connected, then $\operatorname{gr} H$ is commutative by a result of Foissy (Aguiar and Sottile, 2005b, Proposition 1.6 and Remark 1.7).

The Hopf algebra H is *graded* if there is given a vector space decomposition $H = \bigoplus_{m\geq 0} H_m$ such that $\mu(H_p \otimes H_q) \subseteq H_{p+q}$, $\Delta(H_m) \subseteq \bigoplus_{p+q=m} H_p \otimes H_q$, $S(H_m) \subseteq H_m$, $1 \in H_0$, and $\epsilon(H_m) = 0$ for all $m > 0$. In this situation, $H_{(0)} \subseteq H_0$. It follows that if $\dim H_0 = 1$, then H is connected. In this case we say that H is *graded connected* and we have that $H_m \subseteq H_{(m)}$ for all m.

If H is graded, then so is each subspace $H_{(n)}$ with $(H_{(n)})_m = H_{(n)} \cap H_m$. Hence, $\operatorname{gr} H$ inherits a second grading for which $(\operatorname{gr} H)_m$ is the direct sum of the spaces $(H_{(n)})_m\big/(H_{(n-1)})_m$.

2.2 Antipode and Eulerian idempotents

Let H be a connected Hopf algebra. We introduce some notation useful for discussing convolution powers. Put $\Delta^{(0)} = \mathrm{id}$, $\Delta^{(1)} = \Delta$, and $\Delta^{(n)} = (\Delta \otimes \mathrm{id}^{\otimes(n-1)}) \circ \Delta^{(n-1)}$ for all $n \geq 2$. So the superscript is one less than the number of tensor factors in the codomain. Similarly, $\mu^{(n)}$ denotes the map that multiplies $n+1$ elements of H, with $\mu^{(0)} = \mathrm{id}$. Convolution powers of any $P \in \operatorname{End}(H)$ can be written as follows:

$$P^{*0} = \iota\varepsilon \quad \text{and} \quad P^{*n} = \mu^{(n-1)} \circ P^{\otimes n} \circ \Delta^{(n-1)} \ \ (\text{for } n \geq 1).$$

Proposition 1 *Any connected bialgebra is a Hopf algebra with antipode*

$$S = \sum_{k\geq 0} \big(\iota\varepsilon - \mathrm{id}\big)^{*k}. \tag{4}$$

This basic result can be traced back to Sweedler (Sweedler, 1969, Lemma 9.2.3) and Takeuchi (Takeuchi, 1971, Lemma 14); see also Montgomery (Montgomery, 1993, Lem. 5.2.10). It follows by expanding $x^{-1} = \frac{1}{1-(1-x)} = \sum_k (1-x)^k$ in the convolution algebra, with $x = \mathrm{id}$ and $1 = \iota\epsilon$. Connectedness guarantees that the sum in (4) is finite when evaluated on any $h \in H$. More precisely, if $h \in H_{(m)}$, then $(\mathrm{id} - \iota\varepsilon)^{*k}(h) = 0$ for all $k > m$. In particular, this holds if H is graded connected and $h \in H_m$.

We will also need the series expansions of $\log(\mathrm{id})$ in the convolution algebra:

$$\log(\mathrm{id}) = -\sum_{k\geq 1} \frac{1}{k}(\iota\varepsilon - \mathrm{id})^{*k}. \tag{5}$$

Definition 2 To any connected Hopf algebra H are associated *(higher) Eulerian idempotents* $\mathsf{e}^{(k)}$ for $k \geq 0$, given by

$$\mathsf{e}^{(0)} = \iota\varepsilon\,, \quad \mathsf{e}^{(1)} = \log(\mathrm{id})\,, \quad \mathsf{e}^{(k)} = \frac{1}{k!}\big(\mathsf{e}^{(1)}\big)^{*k} \;\; (\text{for } k > 1). \tag{6}$$

The "first" Eulerian idempotent is $\mathsf{e}^{(1)}$. In case H is commutative and cocommutative, the $\mathsf{e}^{(k)}$ form a *complete orthogonal system* of idempotent operators on H. That is,

$$\mathrm{id} = \sum_{k\geq 0} \mathsf{e}^{(k)}, \quad \mathsf{e}^{(k)} \circ \mathsf{e}^{(k)} = \mathsf{e}^{(k)}, \quad \text{and} \quad \mathsf{e}^{(j)} \circ \mathsf{e}^{(k)} = 0 \;\; (\text{for } j \neq k). \tag{7}$$

In addition, if H is cocommutative, $\mathsf{e}^{(k)}$ projects onto the subspace spanned by k-fold products of primitive elements of H (Section 2.1). In particular, $\mathsf{e}^{(1)}$ projects onto $\mathcal{P}(H)$. For proofs of these results, see (Loday, 1992, Ch. 4). It follows from (6) and the identity $x^{*n} = \exp(n\log(x))$ that

$$\mathrm{id}^{*n} = \sum_{k\geq 0} n^k\, \mathsf{e}^{(k)} \quad (\text{for all } n \in \mathbb{Z}). \tag{8}$$

Some instances of these operators in the recent literature include Aguiar and Mahajan (2013), Diaconis et al. (2012), Novelli et al. (2011), and Patras and Schocker (2006). For references to earlier work, see (Aguiar and Mahajan, 2013, §14).

3 Characteristic polynomials for convolution powers

We need two standard results from linear algebra.

Lemma 3 *Fix finite-dimensional spaces $U \subseteq V$, and suppose U is Θ-invariant for some $\Theta \in \mathrm{End}(V)$.*

(i) If $\bar\Theta$ denotes the element of $\mathrm{End}(V/U)$ induced by Θ, and Θ_U denotes the restriction of Θ to U, then the characteristic polynomials of these three maps satisfy $\chi_\Theta(x) = \chi_{\Theta_U}(x)\,\chi_{\bar\Theta}(x)$.

(ii) The characteristic polynomials of Θ and of the dual map $\Theta^* \in \mathrm{End}(V^*)$ are equal. □

We are now ready to prove our main result. From now on we assume that H is a graded connected Hopf algebra for which the homogeneous components H_m are finite-dimensional. We consider the associated graded Hopf algebra $\mathrm{gr}\,H$ and its graded dual $\widetilde{H} = (\mathrm{gr}\,H)^*$. Here $\mathrm{gr}\,H$ is endowed with the grading inherited from that of H (as discussed at the end of Section 2.1), and the dual is with respect to this grading: $\widetilde{H}_m = \big((\mathrm{gr}\,H)_m\big)^*$.

Theorem 4 *For every $n \in \mathbb{Z}$ and $m \in \mathbb{N}$, the characteristic polynomial of $\mathrm{id}^{*n}\big|_{H_m}$ takes the form*

$$\chi\big(\mathrm{id}^{*n}\big|_{H_m}\big) = \prod_{k=0}^{m} (x - n^k)^{\mathsf{eul}(k,m)} \tag{9}$$

for some nonnegative integers $\mathsf{eul}(k,m)$, independent of n. More precisely, we have

$$\mathsf{eul}(k,m) = \dim \mathsf{e}^{(k)}\big(\widetilde{H}_m\big).$$

Moreover, these integers depend only on the graded vector space underlying $\mathcal{P}(\widetilde{H})$.

Corollary 5 *The eigenvalues of the antipode for any graded connected Hopf algebra are* ± 1.

This holds since $S = \mathrm{id}^{*(-1)}$.

Remarks: 1. The previous result fails for general Hopf algebras. Let ω be a primitive cube root of unity and consider Taft's Hopf algebra $T_3(\omega)$ Taft (1971), with generators $\{g, x\}$, and relations $\{g^3 = 1, x^3 = 0, gx = \omega\, xg\}$. The coproduct and antipode are determined by $\Delta(g) = g \otimes g$, $S(g) = g^{-1}$, $\Delta(x) = 1 \otimes x + x \otimes g$, and $S(x) = -xg^{-1}$. Here $x^2 + \omega\, x^2 g$ is an eigenvector of S with eigenvalue ω.

2. Corollary 5 implies that the antipode of a graded connected Hopf algebra is diagonalizable if and only if it is an involution.

3. The antipode of a graded connected Hopf algebra need not be an involution (hence diagonalizable). Take for example the Malvenuto–Reutenauer Hopf algebra, (Aguiar and Sottile, 2005a, Remark 5.6).

Proof of Theorem 4: Since id^{*n} preserves both the grading and the coradical filtration of H, it preserves the filtration

$$(H_{(0)})_m \subseteq (H_{(1)})_m \subseteq \cdots \subseteq (H_{(m)})_m = H_m$$

for each m. By repeated application of Lemma 3(i) we deduce that

$$\chi\big(\mathrm{id}^{*n}\big|_{H_m}\big) = \chi\big(\mathrm{id}^{*n}\big|_{(\mathrm{gr}\, H)_m}\big).$$

The map $\Theta \mapsto \Theta^*$ is an isomorphism of convolution algebras $\mathrm{End}(H) \cong \mathrm{End}(H^*)$ (where duals and endomorphisms are in the graded sense). Together with Lemma 3(ii) this implies that

$$\chi\big(\mathrm{id}^{*n}\big|_{(\mathrm{gr}\, H)_m}\big) = \chi\big(\mathrm{id}^{*n}\big|_{\widetilde{H}_m}\big).$$

Thus, we may work with the cocommutative graded connected Hopf algebra $\widetilde{H}$ instead of H.

In this setting the Eulerian idempotents are available, and from (8) we have that

$$\chi\big(\mathrm{id}^{*n}\big|_{\widetilde{H}_m}\big) = \sum_{k\geq 0} n^k \chi\big(\mathsf{e}^{(k)}\,\big|_{\widetilde{H}_m}\big).$$

It thus suffices to calculate the characteristic polynomial of the $\mathsf{e}^{(k)}$.

Let $\mathfrak{g} = \mathcal{P}(\widetilde{H})$. By CMM, $\widetilde{H} \simeq U(\mathfrak{g})$, and by PBW, $\mathrm{gr}\, U(\mathfrak{g}) \cong S(\mathfrak{g})$. The former is the associated graded Hopf algebra with respect to the coradical filtration of $U(\mathfrak{g})$.

If f and g are filtration-preserving maps, then $\mathrm{gr}(f * g) = (\mathrm{gr}\, f) * (\mathrm{gr}\, g)$. Together with $\mathrm{gr}\, \mathrm{id} = \mathrm{id}$, this implies that $\mathrm{gr}\, \mathsf{e}^{(k)} = \mathsf{e}^{(k)}$, or more precisely, that the following diagram commutes.

$$\begin{tikzcd}
\mathrm{gr}\, U(\mathfrak{g}) \arrow[r, "\mathrm{gr}\, \mathsf{e}^{(k)}"] & \mathrm{gr}\, U(\mathfrak{g}) \\
S(\mathfrak{g}) \arrow[u, "\cong"] \arrow[r, "\mathsf{e}^{(k)}"'] & S(\mathfrak{g}) \arrow[u, "\cong"']
\end{tikzcd}$$

Since by Lemma 3(i) characteristic polynomials (of filtration-preserving maps) are also invariant under gr, we are reduced to computing the characteristic polynomial of $\mathsf{e}^{(k)}$ acting on $S(\mathfrak{g})$.

The action of $\mathsf{e}^{(k)}$ on $S(\mathfrak{g})$ is just projection onto $\mathfrak{g}^k$, the subspace spanned by k-fold products of elements of $\mathfrak{g}$. This follows from the easily verified fact that, for $x_i \in \mathfrak{g}$,

$$\mathrm{id}^{*n}(x_1 \cdots x_k) = n^k x_1 \cdots x_k.$$

It follows that

$$\chi\big(\mathsf{e}^{(k)}\,\big|_{S(\mathfrak{g})_m}\big) = (x - n^k)^{\mathsf{eul}(k,m)},$$

where

$$\mathsf{eul}(k,m) = \dim \mathsf{e}^{(k)}\big(S(\mathfrak{g})_m\big) = \dim\,(\mathfrak{g}^k)_m.$$

and this completes the proof. □

Remark: Since $\mathfrak{g}^k = S^k(\mathfrak{g})$ (the k-th symmetric power of $\mathfrak{g}$), one can be more explicit about the integers $\mathsf{eul}(k,m)$. Let $g_m = \dim \mathfrak{g}_m$ be the dimension of the homogeneous component of degree m of $\mathfrak{g}$. Given a partition λ of the form $\lambda = 1^{k_1} 2^{k_2} \cdots r^{k_r}$, put

$$\binom{\mathfrak{g}}{\lambda} := \binom{g_1 + k_1 - 1}{k_1} \cdots \binom{g_r + k_r - 1}{k_r}.$$

If $|\lambda|$ and $\ell\,(\lambda)$ denote the size and number of parts of λ, respectively, then we have

$$\mathsf{eul}(k,m) = \sum_{\substack{|\lambda|=m \\ \ell(\lambda)=k}} \binom{\mathfrak{g}}{\lambda}. \tag{10}$$

We record an easy corollary to the proof of Theorem 4.

Corollary 6 *If H is graded, connected, then*

$$\mathrm{trace}\big(\mathrm{id}^{*n}\big|_{H_m}\big) = \sum_{k \geq 0} n^k\, \mathsf{eul}(k,m) \tag{11}$$

for all $n \in \mathbb{Z}$ and $m \in \mathbb{N}$. In particular,

$$\mathrm{trace}\big(S\big|_{H_m}\big) = \sum_{k \geq 0} (-1)^k\, \mathsf{eul}(k,m). \tag{12}$$

4 The trace of the antipode and palindromic words

In this section we assume that the graded connected Hopf algebra H is *cofree*. Let $V = \mathcal{P}(H)$. The first observation is that the integers $\mathsf{eul}(k,m)$ in Theorem 4 depend only on the dimensions of the homogeneous components of V. This holds since the dimensions of the homogeneous components of H determine and are determined either by those of V or those of $\mathfrak{g}$. As a result, these two are related by

$$1 - \sum_{n \geq 1} v_n x^n = \prod_{i \geq 1} \big(1 - x^i\big)^{g_i},$$

where $v_n := \dim V_n$ for each positive integer n. This is Witt's formula (Reutenauer, 1993, Cor. 4.14).

Let $\mathsf{pal}(k,m)$ be the number of palindromic words of length k and weight m in an alphabet with v_n letters of weight n. (A word's *length* is its number of letters; its *weight* is the sum of its letters' weights.)

Let $\mathsf{epal}(m)$ and $\mathsf{opal}(m)$ denote the number of palindromes of weight m with even and odd length, respectively. (A palindrome is a word that equals its reversal.) Let h_m be the dimension of H_m, and put $\mathsf{npal}(m) = h_m - \mathsf{epal}(m) - \mathsf{opal}(m)$.

In case H is graded connected and cofree, we have an alternative description of the characteristic polynomial for $S = \mathrm{id}^{*(-1)}$ acting on H_m.

Theorem 7 *In the above situation,*

$$\chi\big(S\big|_{H_m}\big) = (x+1)^{\mathsf{opal}(m)}(x-1)^{\mathsf{epal}(m)}(x^2-1)^{\mathsf{npal}(m)/2}\,. \tag{13}$$

In particular, the trace of the antipode is given by the formula

$$\mathrm{trace}\big(S\big|_{H_m}\big) = \sum_{k=0}^{m}(-1)^k\,\mathsf{pal}(k,m)\,. \tag{14}$$

Proof: By arguments similar to those in used in the proof of Theorem 4, one may take H to be the shuffle algebra $T(V)$, with its canonical Hopf structure. The antipode then acts on a word w in a basis for V by reversing letters: $S(w_1w_2\cdots w_r) = (-1)^r w_r\cdots w_2w_1$ and (13) follows. Finally, note that

$$\mathsf{epal}(m) - \mathsf{opal}(m) = \sum_{k=0}^{m}(-1)^k\,\mathsf{pal}(k,m)$$

to deduce (14) and finish the proof. □

We deduce from (9) and (14) that

$$\sum_{k=0}^{m}(-1)^k\,\mathsf{pal}(k,m) \;=\; \sum_{k=0}^{m}(-1)^k\,\mathsf{eul}(k,m)\,, \tag{15}$$

though these two triangles of integers are generally different.

Example 8 Consider the Malvenuto–Reutenauer Hopf algebra. The alphabet is the set of permutations without global descents. See (Aguiar and Sottile, 2005a, Cor. 6.3) and sequence A003319 in Sloane (OEIS). Looking at the degree three component $\mathfrak{S}Sym_3$, we have

length (k)	1	2	3
permutations	$123, 132, 213$	$231, 312$	321
descent words	$123, 132, 213$	$12\|1, 1\|12$	$1\|1\|1$
$\mathsf{pal}(k,3)$	3	0	1

(Beneath each permutation, we have recorded its expression in terms of letters in the alphabet. On the last line, we count only those words that are palindromic.) The integers $\mathsf{eul}(k,3)$ are computed from (10), where $\mathfrak{g}$ is the free Lie algebra on V and $v(x) = x + x^2 + 3x^3 + 13x^4 + 71x^5 + \cdots$. From Witt's formula, we have $g(x) = x + x^2 + 4x^3 + 17x^4 + 92x^5 + 572x^6 + \cdots$ See A112354 in Sloane (OEIS). So we get $\mathsf{eul}(k,3) = 4, 1, 1$ as $k = 1, 2, 3$.

If we move to the degree four component of $\mathfrak{S}Sym$, one checks that there are 13 permutations without any global descents, and one palindromic permutation with each of 1, 2, and 3 global descents:

$$3412 \equiv 12|12, \qquad 4231 \equiv 1|12|1, \qquad \text{and} \qquad 4321 \equiv 1|1|1|1.$$

Once again, the integers $\mathsf{eul}(k, m)$ are quite different:

$\mathsf{pal}(k, 4)$	13	1	1	1
$\mathsf{eul}(k, 4)$	17	5	1	1

One checks that (15) holds for $m = 3$ and $m = 4$.

5 Schur indicators

A theme occurring in the recent Hopf algebra literature involves a generalization of the *Frobenius–Schur indicator function* of a finite group. If $\rho\colon G \to \mathrm{End}(V)$ is a complex representation of G, then the (second) indicator is

$$\nu_2(G, \rho) = \frac{1}{|G|} \sum_{g \in G} \operatorname{trace} \rho(g^2).$$

The only values this invariant can take are $0, 1, -1$, and this occurs precisely when V is a complex, real or pseudo-real representation, respectively. In Linchenko and Montgomery (2000), a reformulation of the definition was given in terms of convolution powers of the integral[(i)] in $\mathbb{C}G$. This extended the notion of (higher) Schur-indicators to all finite-dimensional Hopf algebras, and has since become a valuable tool for the study of these algebras Kashina et al. (2002); Ng and Schauenburg (2008); Shimizu (2012). In case ρ is the regular representation (and H is semisimple), it is shown in Kashina et al. (2006) that the higher Schur indicators can be reformulated further, removing all mention of the integral:

$$\nu_n(H) = \operatorname{trace}(S \circ \mathrm{id}^{*n}) \quad \text{for} \quad n \geq 0.$$

See also Kashina et al. (2012). These invariants are not well-understood at present. Indeed, the possible eigenvalues of id^{*n} are not even known, much less their multiplicities. Our results lead to the following formula for ν_n in case H is graded, connected (instead of finite-dimensional).

Corollary 9 *If H is a graded connected Hopf algebra, then*

$$\nu_n(H_m) = \sum_{k \geq 0} (-n)^k \,\mathsf{eul}(k, m),$$

where $\mathsf{eul}(k, m)$ is as in Theorem 4.

Proof: As in the proof of Theorem 4, we may assume that H is commutative. Then S is an algebra map, and we have $S \circ \mathrm{id}^{*n} = S \circ \mu^{(n)} \circ \Delta^{(n)} = \mu^{(n)} \circ S^{\otimes n} \circ \Delta^{(n)} = S^{*n}$. Finally, observe that $S^{*n} = \mathrm{id}^{*(-n)}$ and apply Theorem 4. □

(i) A construct present for finite-dimensional Hopf algebras that is unavailable for general graded connected Hopf algebras.

6 Examples and applications

6.1 Symmetric functions

Take $H = \mathit{Sym}$, the Hopf algebra of symmetric functions. On the Schur function basis, the antipode acts by $S(s_\lambda) = (-1)^{|\lambda|} s_{\lambda'}$, where λ' is the partition conjugate to λ. Therefore,

$$\operatorname{trace}\big(S\big|_{H_m}\big) = (-1)^m c(m),$$

where $c(m)$ is the number of self-conjugate partitions of m.

We turn to Corollary 6. For this Hopf algebra, $g_i = 1$ for all $i \geq 1$. Hence $\binom{\mathfrak{g}}{\lambda} = 1$ for all λ, and $\mathsf{eul}(k, m) = p_k(m)$, the number of partitions of m into k parts. From (12) we deduce

$$(-1)^m c(m) = \sum_{k=0}^{m} (-1)^k p_k(m), \tag{16}$$

the identity announced in the introduction. (Note that $p_0(m) = 0$ for $m > 0$.)

We point out that it is possible to obtain this result by considering the power sum basis of *Sym*. Since $S(p_\lambda) = (-1)^{\ell(\lambda)} p_\lambda$, we have

$$\operatorname{trace}(\,S|_{H_m}) = \#\big\{\text{partitions of } m \text{ of even length}\big\} - \#\big\{\text{partitions of } m \text{ of odd length}\big\}.$$

Equating to the former expression for the trace gives (16).

We further illustrate Corollary 6 by deriving certain identities involving the Littlewood–Richardson coefficients $c^{\lambda}_{\mu,\nu}$. Recall that the latter are the structure constants for the product and coproduct on the Schur basis of *Sym*:

$$s_\mu \cdot s_\nu = \sum_{\lambda} c^{\lambda}_{\mu,\nu}\, s_\lambda \qquad \text{and} \qquad \Delta(s_\lambda) = \sum_{\mu,\nu} c^{\lambda}_{\mu,\nu}\, s_\mu \otimes s_\nu .$$

Formula (11) (with $n = \pm 2$) yields the following identities, for all $m \geq 1$:

$$\sum_{\lambda,\mu,\nu \vdash m} (c^{\lambda}_{\mu,\nu})^2 \;=\; \sum_{k=1}^{m} 2^k\, p_k(m) \qquad \text{and} \qquad \sum_{\lambda,\mu,\nu \vdash m} c^{\lambda}_{\mu,\nu} c^{\lambda}_{\mu',\nu'} \;=\; \sum_{k=1}^{m} (-1)^{m-k}\, 2^k\, p_k(m).$$

Note, incidentally, that the fact that the antipode preserves (co)products says that $c^{\lambda}_{\mu,\nu} = c^{\lambda'}_{\mu',\nu'}$.

6.2 Schur P-functions

Let Γ denote the subalgebra of *Sym* generated by the Schur P-functions, P_λ. See (Macdonald, 1995, III.8) for definitions, as well as the results used below. A partition is *strict* if its parts are all distinct. A basis for Γ_m consists of those P_λ with λ a strict partition of m. Let $d(m)$ denote the number of such partitions. For λ strict, $S(P_\lambda) = (-1)^{|\lambda|} P_\lambda$. Therefore,

$$\operatorname{trace}(\,S|_{\Gamma_m}) = (-1)^m d(m).$$

It is well-known that $d(m)$ is also the number of *odd* partitions of m (partitions into odd parts). In fact, Γ is the $\mathbb{Q}$-subalgebra of *Sym* generated by the odd power sums p_{2i+1}, $i \geq 0$. It also follows from this that $\binom{\mathfrak{g}}{\lambda} = 1$ when λ is odd and $\binom{\mathfrak{g}}{\lambda} = 0$ otherwise. Therefore, $\mathsf{eul}(k, m)$ is the number of odd partitions of m of length k. In an odd partition, the parities of m and k are the same. Thus, identity (12) simply counts odd partitions according to their length.

6.3 Quasisymmetric functions

Let us turn to the Hopf algebra $H = \mathit{QSym}$ of quasisymmetric functions, and consider the two standard homogeneous bases for H_m, the *fundamental* F_α and *monomial* M_α quasisymmetric functions, with α a composition of m. The antipode has the following descriptions:

$$S(F_\alpha) = (-1)^m F_{\widetilde{\alpha}'} \quad \text{and} \quad S(M_\alpha) = (-1)^{\ell(\alpha)} \sum_{\beta \leq \alpha} M_{\widetilde{\beta}},$$

where $\widetilde{\gamma}$ is the reversal of the word γ and γ' is the transpose (when drawn as a ribbon-shaped skew-diagram). Note that $\alpha = \widetilde{\alpha}'$ if and only if α is symmetric with respect to reflection across the anti-diagonal (when drawn as a ribbon). There are precisely $2^{(m-1)/2}$ of these when m is odd, and zero when m is even. Calculating the trace on the fundamental basis we thus obtain

$$\operatorname{trace}(\, S|_{H_m}) = \begin{cases} -2^{(m-1)/2} & \text{if } m \text{ is odd,} \\ 0 & \text{otherwise.} \end{cases} \tag{17}$$

The compositions α that contribute to the trace on the monomial basis satisfy $\widetilde{\alpha} \leq \alpha$. Since reversal is an order-preserving involution, this happens if and only if $\widetilde{\alpha} = \alpha$, that is if and only if α is palindromic. Let $\mathsf{pal}(m)$ denote the number of palindromic compositions of m. In m is even, exactly half of the palindromes of length m have odd length; if m is odd, all of them do. We conclude that

$$\operatorname{trace}(\, S|_{H_m}) = \begin{cases} -\,\mathsf{pal}(m) & \text{if } m \text{ is odd,} \\ 0 & \text{otherwise.} \end{cases} \tag{18}$$

Comparing (17) and (18) we deduce that, for all odd m,

$$\mathsf{pal}(m) = 2^{(m-1)/2}.$$

This simple fact can also be deduced by establishing the recursion $\mathsf{pal}(m) = 2\,\mathsf{pal}(m-1)$ for m even and $\mathsf{pal}(m) = \mathsf{pal}(m-1)$ for m odd.

QSym is cofree, so Theorem 7 applies. We have that

$$\mathsf{pal}(k, m) = \begin{cases} \dbinom{\lceil m/2 \rceil - 1}{\lceil k/2 \rceil - 1}, & \text{if } m \text{ is even, or if } m \text{ is odd and } k \text{ is odd,} \\ \quad 0, & \text{if } m \text{ is odd and } k \text{ is even.} \end{cases}$$

Formula (14) boils down in this case to the basic formula $2^h = \sum_{j=0}^{h} \binom{h}{j}$.

6.4 Peak quasisymmetric functions

Let H denote the *peak* Hopf algebra. It is a subalgebra of *QSym*. As *QSym*, it is cofree, and a basis for H_m is indexed by compositions α of m into odd parts. The number of odd compositions of m is the Fibonacci number f_m (with $f_1 = f_2 = 1$).

In Stembridge (1997) and Billera et al. (2003), an analog of the fundamental basis of *QSym* is developed, θ_α. The antipode is $S(\theta_\alpha) = (-1)^m \theta_{\widetilde{\alpha}}$. It follows that

$$\operatorname{trace}(S|_{\Gamma_m}) = \begin{cases} f_{m/2}, & \text{if } m \text{ is even,} \\ -f_{\lceil m/2 \rceil + 1}, & \text{if } m \text{ is odd,} \end{cases}$$

(as palindromic odd compositions of m come from odd compositions of $m/2$). A little more work shows that

$$\mathsf{pal}(k,m) = \begin{cases} \binom{(m+k)/4-1}{(m-k)/4}, & \text{if } m \text{ is even and } 4 \mid (m-k), \\ \binom{\lfloor (m+k-1)/4 \rfloor}{\lfloor (m-k+1)/4 \rfloor}, & \text{if } m \text{ and } k \text{ are odd,} \\ 0, & \text{otherwise.} \end{cases}$$

Formula (14) yields the following basic identities:

$$f_h = \sum_{j=0}^{\lfloor h/2 \rfloor} \binom{h-j-1}{j} \ \text{(for } h \geq 1) \quad \text{and} \quad f_h = \sum_{j=0}^{h-2} \binom{\lfloor (h+j)/2 \rfloor}{\lfloor (h-j)/2 \rfloor} \ \text{(for } h \geq 2).$$

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