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Combinatorial topology of toric arrangements

Giacomo d'Antonio and Emanuele Delucchi

Fachbereich Mathematik und Informatik, Universität Bremen, Bibliothekstraße 1, 28359 Bremen

Abstract. We prove that the complement of a complexified toric arrangement has the homotopy type of a minimal CW-complex, and thus its homology is torsion-free.
To this end, we consider the toric Salvetti complex, a combinatorial model for the arrangement's complement. Using diagrams of acyclic categories we obtain a stratification of this combinatorial model that explicitly associates generators in homology to the "local no-broken-circuit sets" defined in terms of the incidence relations of the arrangement. Then we apply a suitably generalized form of Discrete Morse Theory to describe a sequence of elementary collapses leading from the full model to a minimal complex.

Résumé. On démontre que l'espace complementaire d'un arrangement torique complexifié a le type d'homotopie d'un complexe CW minimal, donc que ses groupes d'homologie sont libres. On considère d'abord un modèle combinatoire du complementaire de l'arrangement: le complexe de Salvetti torique. On obtient une stratification de ce complexe qui fait correspondre explicitement les génerateurs d'homologie aux "circuits-non-rompus locaux" associés aux relations d'incidence de l'arrangement. On applique une forme generalisée de la théorie de Morse discrète pour obtenir une suite de collapsements elementaires qui conduit à un complexe minimale.

Keywords: Combinatorial topology, Toric arrangements, Discrete Morse theory, Torsion-freeness in homology.

1 Introduction

A *toric arrangement* is a finite collection $\mathscr{A} = \{K_1, \ldots, K_n\}$ of level sets of characters of the complex torus, i.e., for all i there is a character $\chi_i \in \mathrm{Hom}((\mathbb{C}^*)^d, \mathbb{C}^*)$ and a 'level' $a_i \in \mathbb{C}^*$ so that $K_i = \chi_i^{-1}(a_i)$.

Toric arrangements play a prominent role in recent work of De Concini, Procesi and Vergne on the link between partition functions and box splines (see e.g. De Concini and Procesi (2010)). A combinatorial framework for this context (in the case where $a_i = 1$ for all i) is given by the theory of arithmetic matroids, studied by D'Adderio and Moci (2011) and Brändén and Moci (2012), leading to nice theoretical constructions and strong enumerative results.

With the aim of improving these enumerative results towards a more structural description, we look at the combinatorial topology of the complement $M(\mathscr{A}) := (\mathbb{C}^*)^d \setminus \bigcup \mathscr{A}$.

We consider the case of complexified toric arrangements, allowing $|a_i| = 1$ for all i. It is known that the Poincaré polynomial of $M(\mathscr{A})$ can be recovered from the associated arithmetic matroid. Moreover, De Concini and Procesi (2005) computed the algebra structure of the cohomology with complex coefficients in the unimodular case. We prove that for any complexified toric arrangement $\mathscr{A}$,

- the space $M(\mathscr{A})$ is *minimal*, i.e., it has the homotopy type of a CW-complex whose cells in every dimension k are counted by the k-th Betti number.
- Hence, the space $M(\mathscr{A})$ is *torsion-free*, that is, the modules $H_k(M(\mathscr{A}),\mathbb{Z}), H^k(M(\mathscr{A}),\mathbb{Z})$ are torsion-free for every k.

The second item is the analogue for toric arrangements of the celebrated theorem by Brieskorn (1971) paving the way for a combinatorial study of the "Orlik-Solomon Algebra" associated to hyperplane arrangements. In this respect, our result is a step towards a "toric Orlik-Solomon algebra".

From a combinatorial point of view, our core data is the *face category* $\mathcal{F}(\mathscr{A})$, which encodes the incidence relations of the induced stratification of the 'real torus' $(S^1)^d \subseteq (\mathbb{C}^*)^d$. As we explain in Section 4, from $\mathcal{F}(\mathscr{A})$ one can construct a combinatorial model for the homotopy type of $M(\mathscr{A})$ which we call toric Salvetti complex because of its relation to the Salvetti complex of a complexified hyperplane arrangement (see Moci and Settepanella (2011); d'Antonio and Delucchi (2011)). A presentation of the fundamental group can also be obtained from $\mathcal{F}(\mathscr{A})$ (d'Antonio and Delucchi (2011)).

We prove minimality by exhibiting a sequence of elementary collapses on the toric Salvetti complex that leads to a minimal complex. To this end, we need to mildly generalize some elements of Discrete Morse Theory in order to be able to work with nonregular CW-complexes or, correspondingly, face categories that are not posets (see Section 5.3). Once this is done, we are left with finding an "acyclic matching" of the face category of the toric Salvetti complex (the so-called *Salvetti category*) with the minimum number of critical cells.

The construction of this matching is the bulk of our work. We use the fact that lower intervals in $\mathcal{F}(\mathscr{A})$ are face posets of real arrangements and call this the "local" structure of $\mathcal{F}(\mathscr{A})$. Correspondingly, the Salvetti category is covered by face posets of Salvetti complexes of these 'local' arrangements.

In Delucchi (2008) it is shown how a particular total ordering of the topes of any oriented matroid leads to a nice stratification of the associated "classical" Salvetti complex with explicitly described strata that each admit a perfect acyclic matching. Here we construct a decomposition of the Salvetti category - indexed by a special total ordering of the "local no-broken-circuits" (see Section 3.1) - which, on each 'local' piece, restricts to the above stratification of the "classical" Salvetti complex. We use diagrams over acyclic categories to prove that every piece of this decomposition is in fact isomorphic to the face category of the stratification of a (smaller dimensional) real torus by a suitable (real) toric arrangement. This construction is explained in Section 5.2.

We are then left to prove that, for any complexified toric arrangement $\mathscr{A}$, the face category $\mathcal{F}(\mathscr{A})$ admits a perfect acyclic matching, as is explained in Section 5.4.

The final step is to patch together the acyclic matchings of the different pieces making sure that they add up to an acyclic matching with the required number of critical cells (Proposition 54).

2 Basics

Definition 1 *Let* $\Lambda \cong \mathbb{Z}^d$ *a finite rank lattice. The corresponding* complex torus *is* $T_\Lambda = \mathrm{Hom}_{\mathbb{Z}}(\Lambda, \mathbb{C}^*)$. *The* compact (or real) torus *corresponding to* Λ *is* $T^c_\Lambda = \mathrm{Hom}_{\mathbb{Z}}(\Lambda, S^1)$*, where* $S^1 := \{z \in \mathbb{C} \mid |z| = 1\}$.

Remark 2 Consider a finite rank lattice Λ and the corresponding torus T_Λ. Every $\lambda \in \Lambda$ defines a *character* of T_Λ, i.e. the function $\chi_\lambda : T_\Lambda \to \mathbb{C}^*$, $\chi_\lambda(\varphi) = \varphi(\lambda)$. Under pointwise multiplication, characters form a lattice which is naturally isomorphic to Λ. Therefore in the following we will identify the character lattice of T_Λ with Λ.

Definition 3 *Consider a finite rank lattice Λ, a* toric arrangement *in T_Λ is a finite set of pairs*

$$\mathscr{A} = \{(\chi_1, a_1), \dots (\chi_n, a_n)\} \subset \Lambda \times \mathbb{C}^*.$$

A toric arrangement $\mathscr{A}$ is called complexified *if $\mathscr{A} \subset \Lambda \times S^1$. Correspondingly, a* real *toric arrangement in T_Λ^c is a finite set of pairs $\mathscr{A}^c = \{(\chi_1, a_1), \dots (\chi_n, a_n)\} \subset \Lambda \times S^1$.*

Remark 4 The abstract definition is clearly equivalent to the one given in the introduction via the canonical isomorphism of Remark 2 and by $K_i := \chi_i^{-1}(a_i)$. Accordingly, we have $M(\mathscr{A}) := T_\Lambda \setminus \bigcup\{K_1, \dots, K_n\}$ and, for a real toric arrangement $\mathscr{A}^c$, $M(\mathscr{A}^c) := T_\Lambda^c \setminus \bigcup\{K_1, \dots, K_n\}$.

Definition 5 *Let Λ be a rank d lattice and let $\mathscr{A}$ be a toric arrangement on T_Λ. The* rank *of $\mathscr{A}$ is rk $(\mathscr{A}) :=$ rk $\langle \chi \mid (\chi, a) \in \mathscr{A} \rangle$. A character $\chi \in \Lambda$ is called* primitive *if, for all $\psi \in \Lambda$, $\chi = \psi^k$ only if $k \in \{-1, 1\}$. The toric arrangement $\mathscr{A}$ is called* primitive *if for each $(\chi, a) \in \mathscr{A}$, χ is primitive. The toric arrangement $\mathscr{A}$ is called* essential *if rk $(\mathscr{A}) = d$.*

Remark 6 For every non primitive arrangement there is a primitive arrangement which has the same complement. Furthermore, if $\mathscr{A}$ is a non essential arrangement, then there exist an essential arrangement $\mathscr{A}'$ such that

$$M(\mathscr{A}) \cong (\mathbb{C}^*)^{d-l} \times M(\mathscr{A}') \text{ where } l = \mathrm{rk}\,(\mathscr{A}').$$

Therefore the topology of $M(\mathscr{A})$ can be derived from the topology of $M(\mathscr{A}')$.

Assumption. From now on we assume every toric arrangement to be *primitive* and *essential*.

2.1 Layers

Let $\mathscr{A} = \{(\chi_1, a_1), \dots, (\chi_n, a_n)\}$ be a toric arrangement on T_Λ. Following De Concini and Procesi (2010) we call *layer* of $\mathscr{A}$ any connected component of a nonempty intersection of some of the subtori K_i (defined in Remark 4). The set of all layers of $\mathscr{A}$ ordered by reverse inclusion is the *poset of layers* of the toric arrangement, denoted by $\mathcal{C}(\mathscr{A})$.

Definition 7 *Let Λ be a finite rank lattice and $\mathscr{A}$ be a toric arrangement in T_Λ. For every sublattice $\Gamma \subseteq \Lambda$ we define the arrangement $\mathscr{A}_\Gamma = \{(\chi, a) \in \mathscr{A} \mid \chi \in \Gamma\}$ and for every layer $X \in \mathcal{C}(\mathscr{A})$ the sublattice $\Gamma_X := \{\chi \in \Lambda \mid \chi$ is constant on $X\} \subseteq \Lambda$. Then, we can define toric arrangements*

$$\mathscr{A}_X := \mathscr{A}_{\Gamma_X} \text{ on } T_{\Gamma_X}, \quad \mathscr{A}^X := \{K_i \cap X \mid X \not\subseteq K_i\} \text{ on the torus } X.$$

Remark 8 Notice that for a layer $X \in \mathcal{C}(\mathscr{A})$ and a hypersurface K of $\mathscr{A}$, the intersection $K \cap X$ is not necessarily connected. In general $K \cap X$ consist of several connected components, each of which is a level set of a character in the torus X. Thus, $\mathscr{A}^X$ is a toric arrangement in the sense of Definition 3.

2.2 Face category

To any complexified toric arrangement is associated the stratification of the real torus T_Λ^c into *chambers* and *faces* induced by the associated ‘real’ arrangement $\mathscr{A}^c$, as follows.

Definition 9 *Consider a complexified toric arrangement* $\mathscr{A} = \{(\chi_1, a_1), \ldots, (\chi_n, a_n)\}$*, its* chambers *are the connected components of* $M(\mathscr{A}^c)$*. We denote the set of chambers of* $\mathscr{A}$ *by* $\mathcal{T}(\mathscr{A})$*.*

The faces *of* $\mathscr{A}$ *are the connected components of the intersections* $\overline{C} \cap X$ *where* $C \in \mathcal{T}(\mathscr{A})$*,* $X \in \mathcal{C}(\mathscr{A})$*. They are the (closed) cells of a polyhedral complex, which we denote by* $\mathcal{D}(\mathscr{A})$*.*

The incidence structure of a (possibly non regular) polyhedral complex X is encoded in a category with one object for every cell, and a morphism for every 'face-relation' among cells. This is called the *face category* of the complex and is denoted by $\mathcal{F}(X)$ (see (d'Antonio and Delucchi, 2011, §2.2.2) for some details on face categories). It is an *acyclic category* in the sense of Kozlov (2008).

Definition 10 *The face category of a complexified toric arrangement* $\mathscr{A}$ *is* $\mathcal{F}(\mathscr{A}) = \mathcal{F}(\mathcal{D}(\mathscr{A}))$*, i.e., the face category of the polyhedral complex* $\mathcal{D}(\mathscr{A})$*.*

2.3 Hyperplane arrangements

Throughout this section let V be a finite dimensional vector space over a field $\mathbb{K}$. An *affine hyperplane* H in V is a level set of a linear functional on V. A set of hyperplanes is called *dependent* or *independent* according to whether the corresponding set of functionals is linearly dependent in V^* or not.

Definition 11 *A* arrangement of hyperplanes *in* V *is a collection* $\mathscr{B}$ *of affine hyperplanes in* V*.*

A hyperplane arrangement $\mathscr{B}$ is called *central* if every hyperplane $H \in \mathscr{B}$ is a linear subspace of V; *finite* if $\mathscr{B}$ is finite; *locally finite* if for every $p \in V$ the set $\{H \in \mathscr{B} \mid p \in H\}$ is finite; *real* (or *complex*) if V is a real (or complex) vector space.

For every central hyperplane arrangement $\mathscr{B}$, the set $\mathcal{L}(\mathscr{B})$ of all nonempty intersections of hyperplanes, ordered by reverse inclusion, is a geometric lattice and defines the *matroid* associated to $\mathscr{B}$.

Definition 12 *An arrangement* $\mathscr{B}$ *in* $\mathbb{C}^d$ *is called* complexified *if every hyperplane* $H \in \mathscr{B}$ *is the complexification of a real hyperplane, i.e., if* $H = \alpha_H^{-1}(a_H)$ *for* $a_H \in \mathbb{R}$ *and* $\alpha_H \in (\mathbb{R}^d)^* \subset (\mathbb{C}^d)^*$*. The real part of a complexified hyperplane arrangement* $\mathscr{B}$ *is* $\mathscr{B}_{\mathbb{R}} = \{H \cap \mathbb{R}^d \mid H \in \mathscr{B}\}$*, an arrangement of hyperplanes in* $\mathbb{R}^d$*.*

A real hyperplane arrangement $\mathscr{B}$ induces a polyhedral decomposition $\mathcal{D}(\mathscr{B})$ of the real ambient space. The face category of this polyhedral complex is denoted $\mathcal{F}(\mathscr{B}) := \mathcal{F}(\mathcal{D}(\mathscr{B}))$. The top cells of this decomposition are called *chambers* of $\mathscr{B}$, the set of chambers is denoted $\mathcal{T}(\mathscr{B})$.

If $\mathscr{B}$ is a complexified hyperplane arrangement, we write $\mathcal{F}(\mathscr{B}) := \mathcal{F}(\mathscr{B}_{\mathbb{R}})$ and $\mathcal{T}(\mathscr{B}) := \mathcal{T}(\mathscr{B}_{\mathbb{R}})$.

2.4 Covering space

The preimage of a toric arrangement $\mathscr{A}$ under the covering map $p : \mathbb{C}^d \cong \mathrm{Hom}_{\mathbb{Z}}(\Lambda, \mathbb{C}) \to \mathrm{Hom}_{\mathbb{Z}}(\Lambda, \mathbb{C}^*) = T_\Lambda,\ \varphi \mapsto \exp \circ \varphi$ is a locally finite affine hyperplane arrangement on $\mathrm{Hom}_{\mathbb{Z}}(\Lambda; \mathbb{C})$. Choosing coordinates we can associate to the character χ_i an integer vector $\alpha_i = \alpha(\chi_i) \in \mathbb{Z}^d$ so that $\chi_i(x) = x_1^{\alpha_{i,1}} \cdots x_d^{\alpha_{i,d}}$ and then let

$$\mathscr{A}^{\upharpoonright} := \{H_{\chi,a'} \mid (\chi, e^{2\pi i a'}) \in \mathscr{A}\} \quad \text{where} \quad H_{\chi,a'} = \{x \in \mathbb{C}^n \mid \langle \alpha(\chi), x \rangle = a'\}.$$

Remark 13 If the toric arrangement $\mathscr{A}$ is complexified, so is the hyperplane arrangement $\mathscr{A}^{\upharpoonright}$.

The lattice Λ acts on $\mathbb{C}^n$ and on $\mathbb{R}^n$ as the group of automorphisms of the covering map p. Consider now the map $q : \mathcal{F}(\mathscr{A}^{\upharpoonright}) \to \mathcal{F}(\mathscr{A})$ induced by p.

Proposition 14 ((d'Antonio and Delucchi, 2011, Lemma 4.8)) *Let $\mathscr{A}$ be a complexified toric arrangement. The map $q : \mathcal{F}(\mathscr{A}^{\upharpoonright}) \to \mathcal{F}(\mathscr{A})$ induces an isomorphism of acyclic categories $\mathcal{F}(\mathscr{A}) \cong \mathcal{F}(\mathscr{A}^{\upharpoonright})/\Lambda$.*

3 Combinatorics

In this section we define local no-broken-circuit sets and prove some combinatorial results about chambers of real hyperplane arrangements.

Lemma 15 *Let $\mathscr{A}$ be a toric arrangement, $X \in \mathcal{C}(\mathscr{A})$ a layer. Then the subposet $\mathcal{C}(\mathscr{A})_{\leq X}$ is the intersection poset of a central hyperplane arrangement $\mathscr{A}[X]$. If $\mathscr{A}$ is complexified, then $\mathscr{A}[X]$ is, too.*

Proof: This is implicit in much of De Concini and Procesi (2005).The proof follows by lifting the layer X to $\mathscr{A}^{\upharpoonright}$. A precise definition of $\mathscr{A}[Y]$ can also be found in Section 5.2.1 below. □

3.1 No-broken-circuit sets, local and global

Recall the terminology of Section 2.3.

Definition 16 *Let $\mathscr{B}$ be a central arrangement of hyperplanes with an arbitrary (but fixed) total order. A* circuit *is a minimal dependent subset $C \subseteq \mathscr{A}$. A* broken circuit *is a subset of the form $C \setminus \{\min C\} \subseteq \mathscr{B}$ obtained from a circuit removing its least element. A* no-broken-circuit set *(or, for short, an* nbc *set) is a subset $N \subseteq \mathscr{B}$ which does not contain any broken circuit. We will write $\mathrm{nbc}(\mathscr{B})$ for the set of no-broken-circuit sets of $\mathscr{B}$ and $\mathrm{nbc}_k(\mathscr{B}) = \{N \in \mathrm{nbc}(\mathscr{B}) \mid |N| = k\}$ for the set of all no-broken-circuit sets of cardinality k.*

Remark 17 For all $k = 0, \ldots, d$, the cardinality $|\mathrm{nbc}_k(\mathscr{B})|$ does not depend on the chosen total ordering.

Definition 18 (De Concini and Procesi (2005)) *Let $\mathscr{A}$ be a toric arrangement of rank d and let us fix a total ordering on $\mathscr{A}$. A* local no-broken-circuit set *of $\mathscr{A}$ is a pair*

$$(X, N) \textit{ with } X \in \mathcal{C}(\mathscr{A}), N \in \mathrm{nbc}_k(\mathscr{A}[X]) \textit{ where } k = d - \dim X$$

We will write $\mathscr{N}$ for the set of local non broken circuits, and partition it into subsets

$$\mathscr{N}_j = \{(X, N) \in \mathscr{N} \mid \dim X = d - j\}.$$

Local no-broken-circuit sets can be used to express the Poincaré polynomial of $M(\mathscr{A})$. The following result was obtained in De Concini and Procesi (2005) by computing de Rham cohomology, in Looijenga (1993) via spectral sequence computations.

Theorem 19 (see (De Concini and Procesi, 2005, Theorem 4.2)) *Consider a toric arrangement $\mathscr{A}$. The Poincaré polynomial of $M(\mathscr{A})$ can be expressed as follows:*

$$P_{\mathscr{A}}(t) = \sum_{j=0}^{\infty} \dim H^j(M(\mathscr{A}); \mathbb{C})\, t^j = \sum_{j=0}^{\infty} |\mathscr{N}_j|\, (t+1)^{k-j}\, t^j.$$

3.2 Combinatorics of real hyperplane arrangements

In this section we will discuss some of the combinatorics of affine arrangements of hyperplanes in real space. Again, we refer the reader to standard references such as Björner et al. (1999); Orlik and Terao (1992) for the basics.

If $\mathscr{B}$ is an arrangement in a real space V, then every hyperplane H is the locus where a linear form $\alpha_H \in V^*$ takes the value a_H. This way we can associate to each $H \in \mathscr{B}$, its *positive* and *negative halfspace*: $H^\epsilon := \{x \in V \mid \operatorname{sgn}(\alpha_H(x) - a_H) = \epsilon\}$ for $\epsilon \in \{+, 0, -\}$.

Definition 20 *Consider a complexified locally finite arrangement $\mathscr{B}$ with any choice of 'sides' for every $H \in \mathscr{B}$. The* sign vector *of a face $F \in \mathcal{F}(\mathscr{B})$ is the function $\gamma_F : \mathscr{B} \to \{-, 0+\}$ defined as: $\gamma_F(H) := \epsilon$ if* $\operatorname{relint} F \subseteq H^\epsilon$.

Notice that chambers are precisely those faces whose sign vector maps $\mathscr{B}$ to $\{-, +\}$.

Definition 21 *Let C_1 and $C_2 \in \mathcal{T}(\mathscr{B})$ be chambers of a real arrangement, and let $B \in \mathcal{T}(\mathscr{B})$ be a distinguished chamber. We will write $S(C_1, C_2) := \{H \in \mathscr{B} \mid \gamma_{C_1}(H) \neq \gamma_{C_2}(H)\}$ for the set of hyperplanes of $\mathscr{B}$ which separate C_1 and C_2.*
For all $C_1, C_2 \in \mathcal{T}(\mathscr{B})$ write $C_1 \le C_2$ if and only if $S(C_1, B) \subseteq S(C_2, B)$. This turns $\mathcal{T}(\mathscr{B})$ into a poset $\mathcal{T}(\mathscr{B})_B$, the poset of regions *of the arrangement $\mathscr{B}$ with base chamber B.*

Remark 22 Let $\mathscr{B}_0$ be a real arrangement and $B \in \mathcal{T}(\mathscr{B}_0)$. Given a subarrangement $\mathscr{B}_1 \subseteq \mathscr{B}_0$, for every chamber $C \in \mathcal{T}(\mathscr{B}_0)$ there is a unique chamber $\widehat{C} \in \mathcal{T}(\mathscr{B}_1)$ with $C \subseteq \widehat{C}$.

Definition 23 *Let $\mathscr{B}_0$ be a real arrangement and let $\succ_0$ denote any total ordering of $\mathcal{T}(\mathscr{B}_0)$. Consider a subarrangement $\mathscr{B}_1 \subseteq \mathscr{B}_0$. The function*

$$\mu[\mathscr{B}_1, \mathscr{B}_0] : \mathcal{T}(\mathscr{B}_1) \to \mathcal{T}(\mathscr{B}_0), \quad C \mapsto \min_{\succ_0}\{K \in \mathcal{T}(\mathscr{B}_0) \mid K \subseteq C\}$$

defines a total ordering $\succ_{0,1}$ on $\mathcal{T}(\mathscr{B}_1)$ by $C \succ_{0,1} D \iff \mu[\mathscr{B}_1, \mathscr{B}_0](C) \succ_0 \mu[\mathscr{B}_1, \mathscr{B}_0](D)$ that we call induced by $\succ_0$.

Proposition 24 (Proposition 11 of d'Antonio and Delucchi (2012)) *Let a base chamber B of $\mathscr{B}_0$ be chosen. If $\succ_0$ is a linear extension of $\mathcal{T}(\mathscr{B}_0)_B$, then $\succ_{0,1}$ is a linear extension of $\mathcal{T}(\mathscr{B}_1)_{\widehat{B}}$.*

4 A combinatorial model for the topology of toric arrangements

In this Section we explain the construction of a combinatorial model for the homotopy type of the complement $M(\mathscr{A})$ of a given complexified toric arrangement.

4.1 The homotopy type of complexified hyperplane arrangements

If $\mathscr{B}$ is a complexified hyperplane arrangement, one can use the combinatorial structure of $\mathscr{B}_{\mathbb{R}}$ to study the topology of $M(\mathscr{B})$. In fact, using combinatorial data about $\mathscr{B}_{\mathbb{R}}$, Salvetti defined a cell complex which embeds in the complement $M(\mathscr{B})$ as a deformation retract (see Salvetti (1987)). We explain this construction.

Definition 25 *Given a face $F \in \mathcal{F}(\mathscr{B})$ and a chamber $C \in \mathcal{T}(\mathscr{B})$, define $C_F \in \mathcal{T}(\mathscr{B})$ as the unique chamber such that*

$$\gamma_{C_F}(H) = \begin{cases} \gamma_F(H) & \textit{if } \gamma_F(H) \neq 0 \\ \gamma_C(H) & \textit{if } \gamma_F(H) = 0 \end{cases}$$

The reader may think of C_F as the one, among the chambers adjacent to F, that "faces" C.

Definition 26 *Consider an affine complexified locally finite arrangement $\mathscr{B}$ and define the* Salvetti poset *as follows:*

$$\mathrm{Sal}(\mathscr{B}) = \{[F, C] \mid F \in \mathcal{F}(\mathscr{B}), C \in \mathcal{T}(\mathscr{B})\, F \leq C\},$$

with the relation $[F_1, C_1] \leq [F_2, C_2] \iff F_2 \leq F_1$ and $(C_2)_{F_1} = C_1$.

Let $\mathscr{B}$ be an affine complexified locally finite hyperplane arrangement. Its Salvetti complex *is the order complex $\mathcal{S}(\mathscr{B}) = \Delta(\mathrm{Sal}(\mathscr{B}))$, i.e., the simplicial complex of all chains.*

Theorem 27 (Salvetti (1987)) *The complex $\mathcal{S}(\mathscr{B})$ is homotopically equivalent to the complement $M(\mathscr{B})$. More precisely $\mathcal{S}(\mathscr{B})$ embeds in $M(\mathscr{B})$ as a deformation retract.*

Remark 28 In fact, the poset $\mathrm{Sal}(\mathscr{B})$ is the face poset of a regular cell complex (of which $\mathcal{S}(\mathscr{B})$ is the barycentric subdivision) whose maximal cells correspond to the pairs $[P, C]$ with $P \in \min \mathcal{F}(\mathscr{B})$, $C \in \mathcal{T}(\mathscr{B})$. It is this complex which is described in Salvetti (1987).

4.2 The toric Salvetti category

In order to define the toric Salvetti category, we need an analogue of Definition 25 for toric arrangements.

Proposition 29 ((d'Antonio and Delucchi, 2011, Proposition 3.12)) *Let Λ be a finite rank lattice, Γ a sublattice of Λ. Let $\mathscr{A}$ a complexified toric arrangement on T_Λ and recall the arrangement $\mathscr{A}_\Gamma$ from Definition 7. The projection $\pi_\Gamma : T_\Lambda \to T_\Gamma$ induces a morphism of acyclic categories $\pi_\Gamma : \mathcal{F}(\mathscr{A}) \to \mathcal{F}(\mathscr{A}_\Gamma)$.*

Consider now a face $F \in \mathcal{F}(\mathscr{A})$. We associate to it the sublattice $\Gamma_F = \{\chi \in \Lambda \mid \chi \text{ is constant on } F\} \subseteq \Lambda$.

Definition 30 *Consider a toric arrangement $\mathscr{A}$ on T_Λ and a face $F \in \mathcal{F}(\mathscr{A})$. The* restriction *of $\mathscr{A}$ to F is the arrangement $\mathscr{A}_F := \mathscr{A}_{\Gamma_F}$ on T_{Γ_F}.*

We will write $\pi_F = \pi_{\Gamma_F} : \mathcal{F}(\mathscr{A}) \to \mathcal{F}(\mathscr{A}_F)$.

Definition 31 ((d'Antonio and Delucchi, 2011, Definition 4.1)) *Let $\mathscr{A}$ be a toric arrangement on the complex torus T_Λ. The* Salvetti category *of $\mathscr{A}$ is the category $\mathrm{Sal}\,\mathscr{A}$ defined as follows.*

(a) The objects are the morphisms in $\mathcal{F}(\mathscr{A})$ between faces and chambers:

$$\mathrm{Obj}(\mathrm{Sal}\,\mathscr{A}) = \{m : F \to C \mid m \in \mathrm{Mor}(\mathcal{F}(\mathscr{A})),\ C \in \mathcal{T}(\mathscr{A})\}.$$

(b) The morphisms are the triples $(n, m_1, m_2) : m_1 \to m_2$, where $m_1 : F_1 \to C_1, m_2 : F_2 \to C_2 \in \mathrm{Obj}(\mathrm{Sal}\,\mathscr{A})$, $n : F_2 \to F_1 \in \mathrm{Mor}(\mathcal{F}(\mathscr{A}))$ and m_1, m_2 satisfy the condition $\pi_{F_1}(m_1) = \pi_{F_1}(m_2)$.

(c) Composition of morphisms is defined as $(n', m_2, m_3) \circ (n, m_1, m_2) = (n \circ n', m_1, m_3)$, whenever n and n' are composable.

Remark 32 The Salvetti category is an acyclic category in the sense of Kozlov (2008).

Definition 33 *Let $\mathscr{A}$ be a complexified toric arrangement; its* Salvetti complex *is the nerve* $\mathcal{S}(\mathscr{A}) = \Delta(\mathrm{Sal}\,\mathscr{A})$.

The following result generalizes (Moci and Settepanella, 2011).

Theorem 34 ((d'Antonio and Delucchi, 2011, Theorem 4.3)) *Let $\mathscr{A}$ be a complexified toric arrangement. The Salvetti complex $\mathcal{S}(\mathscr{A})$ embeds in the complement $M(\mathscr{A})$ as a deformation retract.*

Remark 35 As for the case of affine arrangements, the Salvetti category is the face category of a polyhedral complex, of which the toric Salvetti complex is a subdivision.

For the local structure of the toric Salvetti complex see Remark 40 below.

5 Minimality and torsion-freeness

5.1 'Local' minimality

In the case of complexified arrangements, explicit constructions of a minimal CW-complex for $M(\mathscr{B})$ were given in Salvetti and Settepanella (2007) and in Delucchi (2008). We review the material of (Delucchi, 2008, §4) that will be useful for our later purposes.

Lemma 36 ((Delucchi, 2008, Theorem 4.13)) *Let $\mathscr{B}$ be a central arrangement of real hyperplanes, let $B \in \mathcal{T}(\mathscr{A})$ and let $\preceq$ be any linear extension of the poset $\mathcal{T}(\mathscr{B})_B$. The subset of all $X \in \mathcal{L}(\mathscr{B})$ such that*

$$S(C, C') \cap \mathscr{B}_X \neq \emptyset \quad \text{for all } C' \prec C$$

is an order ideal of $\mathcal{L}(\mathscr{B})$. In particular, it has a well defined and unique minimal element we will call X_C.

Now recall the (cellular) Salvetti complex of Definition 26 and Remark 28. In particular, its maximal cells correspond to the pairs $[P, C]$ where P is a point and C is a chamber. When $\mathscr{B}$ is a central arrangement, the maximal cells correspond to the chambers in $\mathcal{T}(\mathscr{B})$. In this case we can stratify the Salvetti complex assigning to each chamber $C \in \mathcal{T}(\mathscr{B})$ the corresponding maximal cell of $\mathcal{S}(\mathscr{B})$, together with its faces.

Definition 37 *Let $\mathscr{B}$ be a central complexified hyperplane arrangement and write $\min \mathcal{F}(\mathscr{B}) = \{P\}$. Define a stratification of the cellular Salvetti complex $\mathcal{S}(\mathscr{B}) = \bigcup_{C \in \mathcal{T}(\mathscr{B})} \mathcal{S}_C$ through*

$$\mathcal{S}_C := \bigcup \{[F, K] \in \mathrm{Sal}(\mathscr{B}) \mid [F, K] \leq [P, C]\}.$$

Given an arbitrary linear extension $(\mathcal{T}(\mathscr{B}), \preceq)$ of $\mathcal{T}(\mathscr{B})_B$, for all $C \in \mathcal{T}(\mathscr{B})$ define

$$\mathcal{N}_C := \mathcal{S}_C \setminus \left(\bigcup_{D \prec C} \mathcal{S}_D \right), \quad \text{so that} \quad \mathrm{Sal}(\mathscr{B}) = \bigsqcup_{C \in \mathcal{T}(\mathscr{B})} \mathcal{N}_C(\mathscr{B}).$$

Theorem 38 ((Delucchi, 2008, Lemma 4.18)) *There is an isomorphism of posets*

$$\mathcal{N}_C \cong \mathcal{F}(\mathscr{B}^{X_C})^{op}$$

where X_C is the intersection defined via Lemma 36 by the same choice of base chamber and of linear extension of $\mathcal{T}(\mathscr{B})_B$ used to define the subposets $\mathcal{N}_C$, while $\mathscr{B}^{X_C} = \{H \cap X_C \mid H \in \mathscr{B}\}$ denotes the arrangement in the subspace X_C determined by restriction of $\mathscr{B}$.

5.2 Stratification of the toric Salvetti category

We now work our way toward proving the minimality of complements of toric arrangements. We start by defining a stratification of the toric Salvetti complex, in which each stratum corresponds to a local non broken circuit.

5.2.1 Local geometry of complexified toric arrangements

Consider a rank d complexified toric arrangement $\mathscr{A} = \{(\chi_1, a_1), \ldots, (\chi_n, a_n)\}$. Choose coordinates and, as usual, write $\chi_i(x) = x^{\alpha_i}$ for $\alpha_i \in \mathbb{Z}^d$ and $K_i = \{x \in T_\Lambda \mid \chi_i(x) = a_i\}$.

We introduce some central hyperplane arrangements we will work with. Consider the arrangement

$$\mathscr{A}_0 = \{H_i = \ker\langle\alpha_i, \cdot\rangle \mid i = 1, \ldots, n\}$$

in $\mathbb{R}^d$ and, from now on, fix a chamber $B \in \mathcal{T}(\mathscr{A}_0)$ and a linear extension $\prec_0$ of $\mathcal{T}(\mathscr{A}_0)_B$.

Definition 39 *For every face $F \in \mathcal{F}(\mathscr{A})$ and every layer $Y \in \mathcal{C}(\mathscr{A})$ define the arrangements*

$$\mathscr{A}[F] = \{H_i \in \mathscr{A}_0 \mid \chi_i(F) = a_i\}, \qquad \mathscr{A}[Y] = \{H_i \in \mathscr{A}_0 \mid Y \subseteq K_i\},$$

and let $B_F \in \mathcal{T}(\mathscr{A}[F])$, resp. $B_Y \in \mathcal{T}(\mathscr{A}[Y])$, be such that that $B \subseteq B_F$, resp. $B \subseteq B_Y$.

Remark 40 The Salvetti category is the colimit of a diagram over the index category $\mathcal{F}(\mathscr{A})$, which associates to every $F \in \mathcal{F}(\mathscr{A})$ the poset $\mathrm{Sal}(\mathscr{A}[F])$ (d'Antonio and Delucchi, 2012, Lemma 77).

Remark 41 The linear extension $\prec_0$ of $\mathcal{T}(\mathscr{A}_0)_B$ induces, as in Proposition 24, linear extensions $\prec_F$ of $\mathcal{T}(\mathscr{A}[F])_{B_F}$ and $\prec_Y$ of $\mathcal{T}(\mathscr{A}[Y])_{B_Y}$, for every $F \in \mathcal{F}(\mathscr{A})$ and every $Y \in \mathcal{C}(\mathscr{A})$.

Definition 42 *Given $X \in \mathcal{C}(\mathscr{A})$ let $\widetilde{X} \in \mathcal{L}(\mathscr{A}_0)$ be defined as*

$$\widetilde{X} := \bigcap_{X \subseteq K_i} H_i.$$

Definition 43 *Let $Y \in \mathcal{C}(\mathscr{A})$ be a layer of $\mathscr{A}$. For $C \in \mathcal{T}(\mathscr{A}[Y])$ let $X(Y,C) \supseteq Y$ be the layer determined by the intersection defined by Lemma 36 from $\prec_Y$. Analogously, for $C \in \mathcal{T}(\mathscr{A}[F])$ let $X(F,C)$ be defined with respect to $\prec_F$.*

Let then

$$\mathscr{Y} := \{(Y,C) \mid Y \in \mathcal{C}(\mathscr{A}),\ C \in \mathcal{T}(\mathscr{A}[Y]),\ X(Y,C) = Y\}.$$

Moreover, for $i = 0, \ldots, d$ let $\mathscr{Y}_i := \{(Y,C) \in \mathscr{Y} \mid \dim(Y) = i\}$.

Lemma 44 *Let $\mathscr{A}$ be a rank d toric arrangement. For all $i = 0, \ldots d$, we have $|\mathscr{Y}_i| = |\mathscr{N}_i|$.*

As a last preparation, we need to be able to map morphisms $m : F \to G$ of $\mathcal{F}(\mathscr{A})$ to the corresponding face of the arrangement $\mathscr{A}[F]$.

Definition 45 *Consider a toric arrangement $\mathscr{A}$ on $T_\Lambda \cong (\mathbb{C}^*)^k$ and a morphism $m : F \to G$ of $\mathcal{F}(\mathscr{A})$. We associate to m a face $F_m \in \mathcal{F}(\mathscr{A}[F])$ as follows.*

First, fix an $F^\upharpoonright \in \mathcal{F}(\mathscr{A}^\upharpoonright)$ such that $q(F^\upharpoonright) = F$. From Proposition 29 and from the freeness of the action of Λ it follows that there is a unique $G^\upharpoonright \in \mathcal{F}(\mathscr{A}^\upharpoonright)$ such that $q(F^\upharpoonright \leq G^\upharpoonright) = m$. Then, consider the arrangement $\mathscr{A}^\upharpoonright_{F^\upharpoonright} = \{H \in \mathscr{A}^\upharpoonright \mid F^\upharpoonright \in H\}$. Clearly, up to translation, $\mathscr{A}^\upharpoonright_{F^\upharpoonright} = \mathscr{A}[F]$ and we can identify the two arrangements. Now define F_m as the face of $\mathscr{A}[F]$ which contains $G^\upharpoonright$. That is, in terms of sign vectors and identifying each $H \in \mathscr{A}[F]$ with its unique translate which contains $G^\upharpoonright$: $\gamma_{F_m} = \gamma_{G^\upharpoonright|\mathscr{A}[F]}$. In particular, when G is a chamber, then F_m also is.

5.2.2 Definition of the strata

Definition 46 *Recall Definition 23. The assignment $(Y, C) \mapsto \mu[\mathscr{A}[Y], \mathscr{A}_0](C)$ defines a function $\xi_0 : \mathscr{Y} \to \mathcal{T}(\mathscr{A}_0)_B$. Choose, and fix, a total order $\dashv$ on $\mathscr{Y}$ that makes this function order preserving.*

Definition 47 *Define the map $\theta : \mathrm{Sal}(\mathscr{A}) \to \mathscr{Y};\ (m : F \to C) \mapsto (X(F, F_m),\ \sigma_{\mathscr{A}[X(F,F_m)]}(F_m))$.*

Through θ we can now define a filtration of $\mathrm{Sal}(\mathscr{A})$.

Definition 48 *Given a complexified toric arrangement $\mathscr{A}$ on $(\mathbb{C}^*)^d$, we consider the following stratification of Sal($\mathscr{A}$) indexed by $\mathscr{Y}$: we write Sal($\mathscr{A}$) $= \cup_{(Y,C)\in\mathscr{Y}} \mathcal{S}_{(Y,C)}$ where $\mathcal{S}_{(Y,C)}$ is the induced subcategory with $\mathrm{Ob}(\mathcal{S}_{(Y,C)}) = \{m \in \mathrm{Ob}(\mathrm{Sal}(\mathscr{A})) \mid \exists (m \to n) \in \mathrm{Mor}(\mathrm{Sal}(\mathscr{A})),\ n \in \theta^{-1}(Y, C)\}$. Moreover, recall the total ordering $\vdash$ on $\mathscr{Y}$ and define*

$$\mathcal{N}_y := \mathcal{S}_y \setminus \bigcup_{y' \dashv y} \mathcal{S}_{y'}.$$

We now come to the gist of our construction: everything has been arranged so that every stratum, as a category, is isomorphic to the face category of a real toric arrangement.

Theorem 49 *Consider a complexified toric arrangement $\mathscr{A}$ and for $(Y, C) \in \mathscr{Y}$ let $\mathcal{N}_{(Y,C)}$ be as in Definition 48. Then there is an isomorphism of acyclic categories*

$$\mathcal{N}_{(Y,C)} \cong \mathcal{F}(\mathscr{A}^Y)^{op}.$$

The details of the proof are very technical and quite lengthy. We believe that it is in the best interest of the clarity of this extended abstract to refer the interested reader to the full treatment given in d'Antonio and Delucchi (2012).

5.3 Discrete Morse Theory for acyclic categories

Our proof of minimality will consist in describing a sequence of cellular collapses on the toric Salvetti complex, which is not necessarily a regular cell complex. We need thus to extend discrete Morse theory from posets to acyclic categories. The setup used in the textbook of Kozlov (2008) happens to lend itself very nicely to such a generalization - in fact, once the right definitions are made, even the proofs given there just need some minor additional observation.

We will omit the technicalities in this extended abstract, and refer to (d'Antonio and Delucchi, 2012, §3) for a more detailed account. We will only say that the notion of acyclic matching extends easily to acyclic categories so that an acyclic matching on the face category of a CW-complex X defines a sequence of cellular collapses on X that preserve the homotopy type and leads to a complex with as many cells in each dimension as there are corresponding critical (unmatched) cells in the original matching (d'Antonio and Delucchi, 2012, Definition 50, Theorem 53). We will call an acyclic matching *perfect* if its number of critical cells in dimension k is the k-th Betti number of X.

Moreover, the well-known Patchwork Lemma (Kozlov, 2008, Theorem 11.10) generalizes.

Lemma 50 ("Patchwork Lemma", Lemma 52 of d'Antonio and Delucchi (2012)) *Consider a functor of acyclic categories $\varphi : \mathcal{C} \to \mathcal{C}'$ and suppose that for each object c of $\mathcal{C}'$ an acyclic matching $\mathfrak{M}_c$ of $\varphi^{-1}(c)$ is given. Then the matching $\mathfrak{M} := \bigcup_{c \in \mathrm{Ob}\,\mathcal{C}'} \mathfrak{M}_c$ of $\mathcal{C}$ is acyclic.*

5.4 Perfect acyclic matchings for compact tori

Let $\mathscr{A}$ be a complexified toric arrangement in T_Λ and let $(\chi_1, a_1), \ldots, (\chi_d, a_d) \in \mathscr{A}$ be such that $\alpha_1, \ldots, \alpha_d$ (see Section 2.4) are ($\mathbb{Q}$-) linearly independent. Then $P = \cap_i K_i \in \max \mathcal{C}(\mathscr{A})$. Up to a biholomorphic transformation we may suppose that P is the origin of the torus. For $i = 1, \ldots, d$ let H_i^1 denote the hyperplane of $\mathscr{A}^\upharpoonright$ lifting K_i at the origin of $\mathrm{Hom}(\Lambda, \mathbb{R}) \simeq \mathbb{R}^d$. We identify for ease of notation $\Lambda \simeq \mathbb{Z}^d \subseteq \mathbb{R}^d$, and in particular think of α_i as the normal vector to H_i^1.

For $j \in [d]$ we consider the rank $j-1$ lattice $\Lambda_j := \mathbb{Z}^d \cap \bigcap_{i \geq j} H_i^1$. It is a standard exercise in algebra to find a basis $u_1, \ldots, u_d$ of Λ such that for all $i = 1, \ldots, d$, the elements $u_1, \ldots, u_{i-1}$ are a basis of Λ_i.

In particular, $u_i \notin H_i^1$, hence $u_i(H_i^1) \neq H_i^1$. Moreover, without loss of generality we may suppose $u_i \in (H_i^1)^+ := \{x \in \mathbb{R}^d \mid \langle x, \alpha_i \rangle \geq 0\}$.

For $i = 1, \ldots, d$, let $(H_i^2)^+ := u_i((H_i^1)^+)$, and define $Q := \bigcap_{i=1}^d [(H_i^1)^+ \setminus (H_i^2)^+]$.

Then, Q is a fundamental region for the action of Λ on $\mathbb{R}^d$ (d'Antonio and Delucchi, 2012, Lemma 86).

Definition 51 *Let $\mathscr{A}$ be a rank d toric arrangement, and let $\mathcal{B}_d$ be the 'boolean poset on d elements', i.e., the acyclic category on the subsets of $[d]$ with the inclusion morphisms. Since $\mathcal{B}_d$ is a poset, the function $\mathrm{Ob}(\mathcal{F}(\mathscr{A})) \to \mathrm{Ob}(\mathcal{B}_d)$, $F \mapsto \{i \in [d] \mid F \subseteq K_i\}$, induces a well defined functor of acyclic categories $\mathcal{I} : \mathcal{F}(\mathscr{A}) \to \mathcal{B}_d^{op}$.*

For every $I \subseteq [d]$ define the category $\mathcal{F}_I := \mathcal{I}^{-1}(I)$.

Lemma 52 (Lemma 89 of d'Antonio and Delucchi (2012)) *For all $I \subseteq [d]$, the subcategory $\mathcal{F}_I$ is a poset admitting an acyclic matching with only one critical element (in top rank).*

Proposition 53 *For any complexified toric arrangement $\mathscr{A}$, the acyclic category $\mathcal{F}(\mathscr{A})$ admits a perfect acyclic matching.*

Proof: Let $\mathscr{A}$ be of rank d. The proof is a straightforward application of the Patchwork Lemma (Lemma 50) in order to merge the 2^d acyclic matchings described in Lemma 52 along the map $\mathcal{I}$ of Definition 51. The resulting 'global' acyclic matching has 2^d critical elements and is thus perfect. □

5.5 Minimality

Let $\mathscr{A}$ be a (complexified) toric arrangement.

Proposition 54 *The Salvetti category* $\mathrm{Sal}\,\mathscr{A}$ *admits a perfect acyclic matching.*

Proof: Let P denote the acyclic category given by the $|\mathscr{Y}|$-chain. We define a functor of acyclic categories

$$\varphi : \mathrm{Sal}\,\mathscr{A} \to P; \quad m \mapsto (Y, C) \text{ for } m \in \mathcal{N}_{(Y,C)}$$

and with Theorem 49 we have an isomorphism of acyclic categories $\varphi^{-1}((Y,C)) = \mathcal{N}_{(Y,C)} \simeq \mathcal{F}(\mathscr{A}_Y)$. Then, by Proposition 53, $\varphi^{-1}((Y,C))$ has an acyclic matching with $2^{d-\mathrm{rk}\,X}$ critical cells.

An application of the Patchwork Lemma 50 gives then an acyclic matching on $\mathrm{Sal}(\mathscr{A})$ with $\sum_j |\mathscr{Y}_j| 2^{d-j} = \sum_j |\mathcal{N}_j| 2^{d-j} = P_{\mathscr{A}}(1)$ critical cells, where the first equality is given by Lemma 44. This matching is thus perfect. □

Corollary 55 *The complement $M(\mathscr{A})$ is a minimal space.*

Corollary 56 *The groups $H_k(M(\mathscr{A}), \mathbb{Z})$, $H^k(M(\mathscr{A}), \mathbb{Z})$ are torsion free for all k.*

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