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Optimal $L(h,k)$-Labeling of Regular Grids[†]

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The $L(h,k)$-labeling is an assignment of non negative integer labels to the nodes of a graph such that 'close' nodes have labels which differ by at least k, and 'very close' nodes have labels which differ by at least h. The span of an $L(h,k)$-labeling is the difference between the largest and the smallest assigned label. We study $L(h,k)$-labelings of cellular, squared and hexagonal grids, seeking those with minimum span for each value of k and $h \geq k$. The $L(h,k)$-labeling problem has been intensively studied in some special cases, i.e. when $k = 0$ (vertex coloring), $h = k$ (vertex coloring the square of the graph) and $h = 2k$ (radio- or λ -coloring) but no results are known in the general case for regular grids. In this paper, we completely solve the $L(h,k)$-labeling problem on cellular grids, finding exact values of the span for each value of h and k; only in a small interval we provide different upper and lower bounds. For the sake of completeness, we study also hexagonal and squared grids.

Keywords: $L(h,k)$-labeling, cellular grids, triangular grids, hexagonal grids, squared grids.

1 Introduction

The $L(h,k)$-*labeling problem* consists in assigning non negative integer labels to the nodes of a graph such that nodes at distance two have labels which differ by at least k, and adjacent nodes have labels which differ by at least h. The *span* of an $L(h,k)$-labeling is the difference between the largest and the smallest assigned frequency. The aim of the $L(h,k)$-labeling problem is to satisfy the distance constraints using the minimum span. This graph theoretical problem arises from the problem of assigning frequencies to the transcievers of a wireless network in order to avoid some kinds of interference (i.e. direct and hidden collision); in this setting, the nature of the environment and the geographical distance are the major factors determining parameters h and k, and it is usually assumed $h \geq k$. Since its formal definition (18) the $L(h,k)$-labeling problem has been widely studied by means of techniques from disparate research areas and receiving many names (for a survey see (5)). However, almost all the literature concerns the special case of $k = 1$ and $h = 2$ or $h = 1$, and very few papers (8; 14; 15; 16; 20) investigate on the more general problem. Nevertheless, the solution of the problem for any h and k is worthy since it allows one to handle more realistic scenarios. Observe that, when $k = 0$, for any fixed h, the problem is equivalent to the classical vertex coloring problem, and when $h = k$ it becomes the problem of optimally coloring

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the nodes of the square of the input graph; finally, when $h = 2k$ the problem has been called radio- or λ-coloring problem. All these problems have been intensively studied.

The decisional version of the $L(h, k)$-labeling problem is NP-complete even for small values of h and k (2). This motivates seeking optimal solutions on particular classes of graphs.

In this paper, we completely solve the $L(h, k)$-labeling problem on cellular grids, finding exact values of the span for each value of h and k; only in a small interval we provide different upper and lower bounds. For the sake of completeness, we study also hexagonal and squared grids.

Exploiting the upper bounds presented in this paper, a label can be assigned to any node in a distributed fashion in constant time in all considered grids, provided that the relative position of the node in the graph is locally known. In this paper, the presented upper bounds will be described by means of formulas determining the color of a node as function of its own coordinates; nevertheless, figures will help to have the intuition of the labeling schemes.

2 Preliminaries and Discussion of the Results

For any non negative real values k and $h \geq k$, an $L(h, k)$-*labeling* of a graph $G = (V, E)$ is a function $L : V \rightarrow \mathbb{R}$ such that

- $|L(u) - L(v)| \geq h$ if $(u, v) \in E$ and
- $|L(u) - L(v)| \geq k$ if there exists $w \in V$ such that $(u, w) \in E$ and $(w, v) \in E$.

The *span* of an $L(h, k)$-labeling is the difference between the largest and the smallest value of L, so it is not restrictive to assume 0 as the smallest value of L. We denote by $\lambda_{h,k}(G)$ the smallest integer λ such that graph G has an $L(h, k)$-labeling of span λ.

In this paper, we consider the infinite cellular hexagonal and squared grids, where the position of each node is defined by a couple of integer coordinates, as shown in Fig. 1. Given a certain node (x, y) in a cellular grid, its neighbors are $(x+1, y)$, $(x-1, y)$, $(x, y-1)$, $(x, y+1)$, $(x-1, y-1)$, $(x+1, y+1)$. The nodes at distance 2 from (x, y) are $(x+2, y)$, $(x-2, y)$, $(x, y+2)$, $(x, y-2)$, $(x+2, y+1)$, $(x-2, y-1)$, $(x+1, y-1)$, $(x-1, y+1)$, $(x+2, y+2)$, $(x-2, y-2)$, $(x+1, y+2)$ and $(x-1, y-2)$.

The reader can easily determine the neighbors of (x, y) and the nodes at distance 2 from (x, y) in a hexagonal and squared grid (highlighted by grey areas in Fig. 1.)

We will use these sets of nodes to prove that the presented labelings are feasible. For the regularity of the grids, it is not restrictive to consider only nodes whose coordinates are lexicographically greater than (x, y) (otherwise it is enough to swap the role of the nodes).

In this paper we study the $L(h, k)$-labeling problem on the cellular grid C, proving that:

$$\begin{array}{l} 2h + 4k \leq \lambda_{h,k}(C) \leq \min(6h, 8k) \text{ if } k \leq h \leq 2k; \\ 3h + 2k \leq \lambda_{h,k}(C) \leq \min(4h, 11k) \text{ if } 2k \leq h \leq 3k; \\ \lambda_{h,k}(C) = 3h + 2k \text{ if } 3k \leq h < 4k; \\ \lambda_{h,k}(C) = 2h + 6k \text{ if } h \geq 4k. \end{array}$$

For the sake of completeness we study also the hexagonal grid H, showing that:

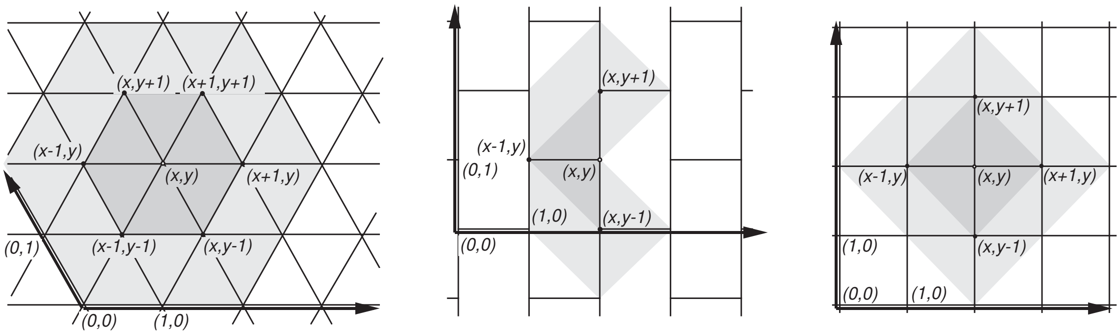


Fig. 1: Cellular, hexagonal and squared grids, where the nodes at distance 1 and 2 from the general node (x, y) are higlighted. Observe that in the hexagonal grid the coordinates of these nodes change according to the parity of x.

$2h + k \leq \lambda_{h,k}(H) \leq \min(3h, 5k)$ if $k \leq h \leq 2k$;
$\lambda_{h,k}(H) = 2h + k$ if $2k \leq h < 3k$;
$\lambda_{h,k}(H) = h + 4k$ if $h \geq 3k$;

Finally, we improve the results by Georges and Mauro (13) achieved as a special case of the more general $L(h, k)$-labeling problem on product of paths. Here we state only the results achieved in this paper; see Figure 2(a) for the complete results.

$2h + 2k \leq \lambda_{h,k}(S) \leq \min(4h, 2h + 3k - 1, 6k)$ if $k \leq h \leq 2k$;
$2h + 2k \leq \lambda_{h,k}(S) \leq \min(3h, 2h + 3k - 1, 8k)$ if $2k \leq h \leq 3k$.

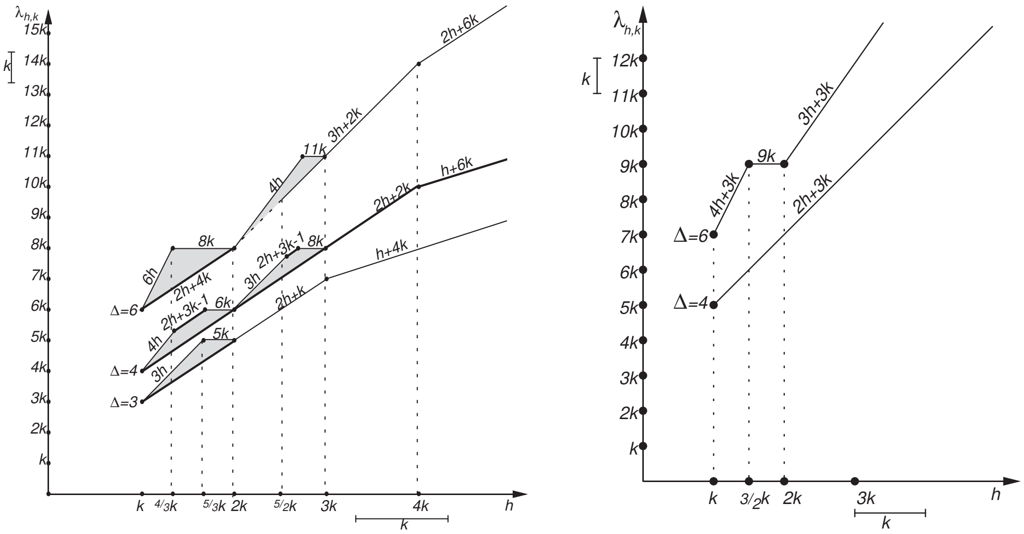
The really important parameter is the ratio h/k. This is the reason why, in the graphical summary of results depicted in Fig. 2(a), h is a function of k.

The $L(h, k)$-labeling problem on regular grids has already been studied in (3) for $h = 2$ and $k = 1$, and in (6) for $h = 0, 1, 2$ and $k = 1$. Of course, the results obtained in this paper include as special case the previous ones. Also Griggs and Jin (17) have independently studied the same problem using completely different techniques.

In (20) the distance between two labels $i, j \in \{0, 1, \ldots, n-1\}$ is defined as *min*$\{|i-j|, n-|i-j|\}$. Using this definition and restricting h and k to be integer, the authors study a variant of $L(h, k)$-labeling on triangular and squared grids (for a summary of their results see Fig. 2(b)). We will call $L^c(h, k)$-labeling problem this variant. The authors of (20) approach the $L^c(h, k)$-labeling problem from a purely combinatorial point of view, with completely different techniques, for each integer h and k. Furthermore, observe that – despite the similarity of $L(h, k)$- and $L^c(h, k)$-labeling problems – it does not seem possible to shift from results in (20) to ours (compare Fig. 2(a) and Fig. 2(b)).

Before proving one by one all bounds listed above, we state some general results that will be useful in the following.

Theorem 2.1 *(13) Given any regular grid of the plane G with degree Δ ($\Delta = 3, 4$ or 6), the following conditions for $\lambda_{h,k}(G)$ hold:*



(a) Summary of the results: grey areas denote gaps between the upper and the lower bounds. Bold lines represent already known results.

(b) Summary of the upper bounds for $\lambda^c_{h,k}$ presented in (20), where h and k are integer values.

Fig. 2: Summaries of bounds.

a. $\lambda_{h,k}(G) \geq 2h + (\Delta - 2)k$ *for any* $k \leq h \leq \Delta k$*;*

b. $\lambda_{h,k}(G) \geq h + 2(\Delta - 1)k$ *for any* $h \geq \Delta k$*.*

Thanks to the generality of its statement, Theorem 2.1 will be exploited in the following in order to obtain immediate lower bounds on $\lambda_{h,k}$. Before concluding this section, we depict in Fig. 3 optimal $L(1,1)$-, $L(2,1)$- and $L(3,1)$-labelings of the regular grids that will be used in the following.

3 Cellular Graphs

Given a cellular grid with an optimal $L(h,k)$-labeling, for any node x we call $a_1, a_2, \ldots a_6$ its neighbors arranged around x (see Fig. 4). It is not restrictive to assume that a_1 has the smallest label, and that $L(a_2) < L(a_6)$.

In this section, we derive exact values of $\lambda_{h,k}(C)$ by proving coinciding upper and lower bounds, except for interval $k \leq h \leq 3k$, where bounds are slightly different.

3.1 $k \leq h \leq 2k$

Theorem 3.1 *If* $k \leq h \leq 2k$*, then* $2h + k \leq \lambda_{h,k}(C) \leq$ *min* $(6h, 8k)$.

Proof:

Lower bound. It directly descends from Theorem 2.1 a

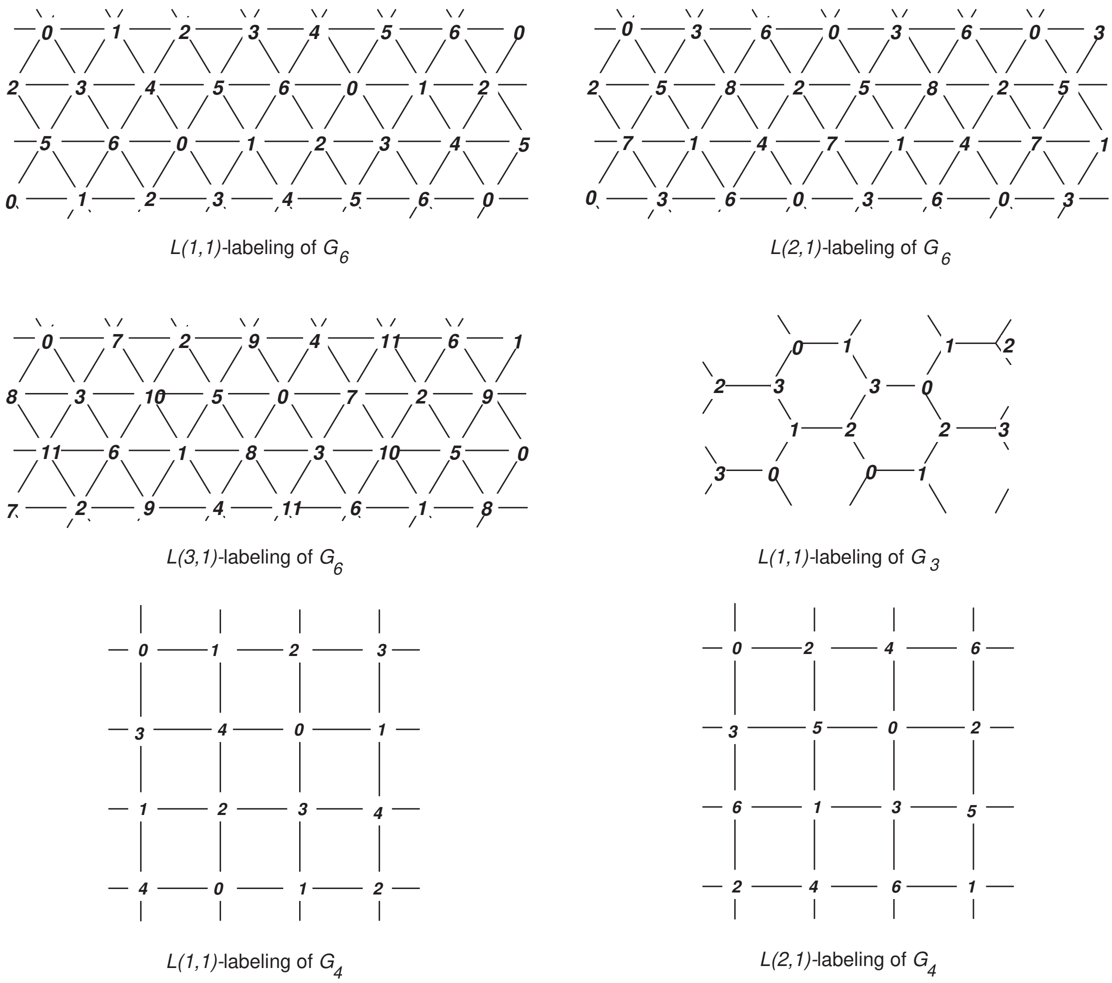


Fig. 3: $L(1,1)$-, $L(2,1)$- and $L(3,1)$-labelings of regular grids.

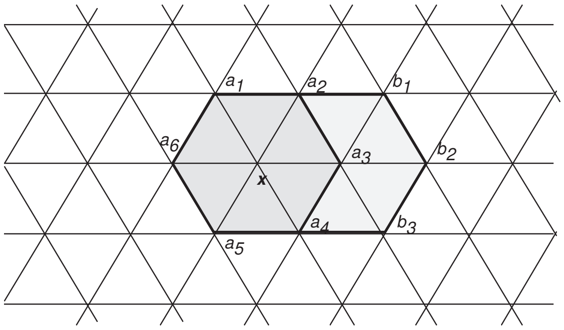


Fig. 4: A general node x and all its neighbors in a cellular graph.

Upper bound. Given any node (x, y) of the cellular grid, consider the following labeling function (see Fig. 5.a):

$$L((x, y)) = ((x + 4y) \text{mod } 7)h.$$

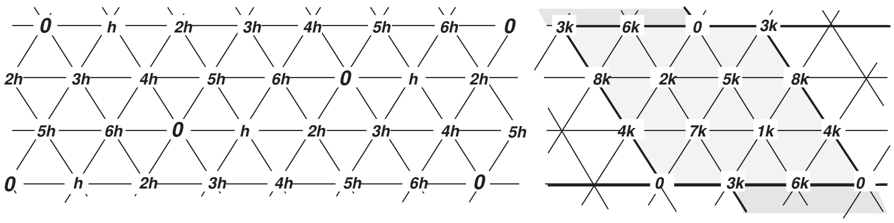
This labeling is feasible, indeed $|L((x + 1, y)) - L((x, y))| \geq h$, $|L((x, y + 1)) - L((x, y))| \geq 3h$ and $|L((x + 1, y + 1)) - L((x, y))| \geq 2h$, so the distance 1 constraint is always respected. Analogously, the distance 2 constraint is respected, too; indeed: $|L((x + 2, y)) - L((x, y))| \geq 2h$, $|L((x, y + 2)) - L((x, y))| \geq h$, $|L((x+2, y+1)) - L((x, y))| \geq h$, $|L((x+1, y-1)) - L((x, y))| \geq 3h$, $|L((x+2, y+2)) - L((x, y))| \geq 3h$ and $|L((x + 1, y + 2)) - L((x, y))| \geq 2h$, i.e. the minimum distance between $L((x, y))$ and the label of any node at distance 2 from (x, y) is at least $h \geq k$.

The span of the presented labeling is $6h$. Observe that the resulting labeling is essentially identical to an optimal $L(1, 1)$-labeling of the cellular grid, where all values are multiplyied by h.

Consider now the following labeling function (see Fig. 5.b):

$$L((x, y)) = ((3x + 4y) \text{mod } 9)k.$$

Also this labeling is feasible: $|L((x + 1, y)) - L((x, y))| \geq 3k$, $|L((x, y + 1)) - L((x, y))| \geq 4k$ and $|L((x + 1, y + 1)) - L((x, y))| \geq 2k \geq h$. Analogously, the distance between $L((x, y))$ and the label of any node at distance 2 from (x, y) is always $\geq k$. The span of the presented labeling is $8k$.



a. b.

Fig. 5: Feasible labelings of a cellular graph when a. $k \leq h \leq \frac{4}{3}k$ and when b. $\frac{4}{3}k \leq h \leq 2k$.

Observe that this labeling is exactly the same as an optimal $L(2,1)$-labeling, where each value has been multiplied by k. It follows that, when $k \leq h \leq 2k$, $\lambda_{h,k}(C) \leq \min\,(6h, 8k)$. Combining the results for the two labelings, it follows that, when $k < h \leq \frac{5}{3}k$ then $\lambda_{h,k}(C) \leq 6h$, and when $\frac{5}{3}k \leq h \leq 2k$ then $\lambda_{h,k}(C) \leq 8k$. □

3.2 $2k \leq h < 4k$

Theorem 3.2 *If $3k < h < 4k$, then $\lambda_{h,k}(C) = 3h + 2k$; if $2k \leq h \leq 3k$ then $3h + 2k \leq \lambda_{h,k}(C) \leq min(4h, 11k)$.*

Proof:

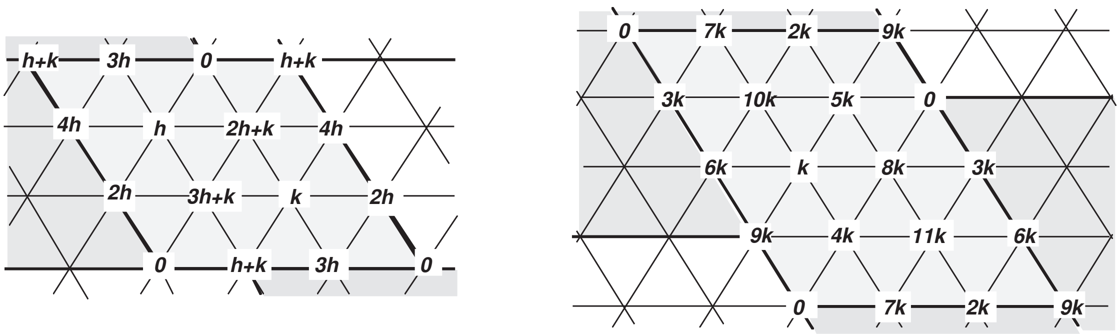
Upper bound. If $2k \leq h \leq 3k$, consider the two following labeling functions (see Figs. 6.a and 6.b):

$$L((x,y)) = \lfloor((3x+4y)\text{mod } 9)/2\rfloor h + ((3x+4y)\text{mod } 9)\text{mod } 2)k$$

and

$$L((x,y)) = ((7x+9y)\text{mod } 12)k.$$

Analogously to the previous proofs, it is easy to check that both labelings are feasible, comparing $L((x,y))$ with the label of all nodes at distance 1 and 2 from (x,y). Furthermore, the span of the presented labelings are $4h$ and $11k$, respectively. It follows that $\lambda_{h,k}(C) \leq \min(4h, 11k)$. Combining the results for the two labelings, it follows that when $2k < h \leq \frac{11}{4}k$ then $\lambda_{h,k}(C) \leq 4h$ and when $\frac{11}{4}k \leq k < 3k$ then $\lambda_{h,k}(C) \leq 11k$. Observe that also the labeling in Fig. 6.a can be obtained from an optimal $L(2,1)$-labeling by the following substitutions: $(0,0), (1,k), (2,h), (3,h+k), (4,2h), (5,2h+k), (6,3h), (7,3h+k)$ and $(8,4h)$, while the labeling in Fig. 6.b can be obtained from an optimal $L(3,1)$-labeling multiplying each value by k.



a. b.

Fig. 6: Two feasible labelings of a cellular graph when $2k \leq h \leq 3k$. Their span is a. $4h$ and b. $11k$.

If $3k < h < 4k$ consider the labeling function defined by the following formula (see Fig. 7.a):

$$L((x,y)) = ((y \text{ mod } 4 + x \text{ mod } 3)\text{mod } 4)h + (x \text{ mod } 3)k.$$

The produced coloring is a feasible $L(h,k)$-labeling and its span is $3h + 2k$.

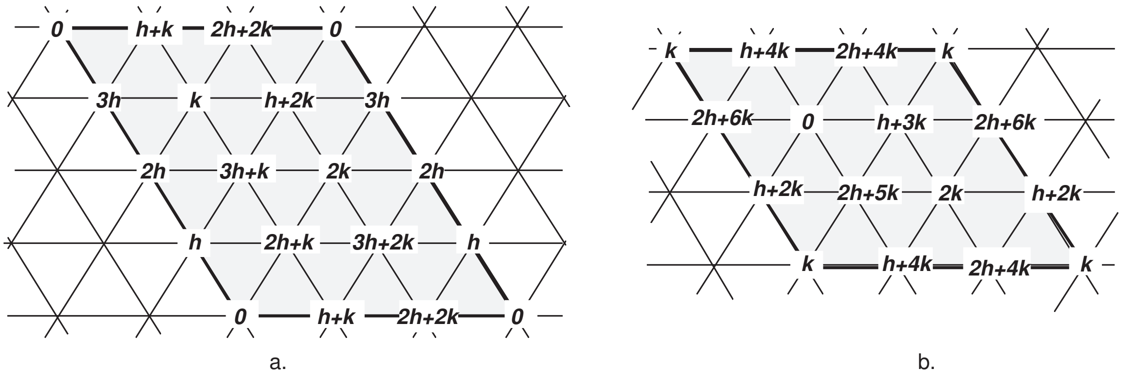


Fig. 7: Two optimally labeled portions of cellular graph when a. $3k < h < 4k$ and b. $h \geq 4k$.

Lower bound. Let be given a cellular graph with any optimal $L(h, k)$-labeling, $2k \leq h < 4k$. We consider a node a and all possible relative orders of the 7 distinct labels of a and its neighbors.

The proof is based on a systematic way of describing all the different cases in which a labeling of smaller span could be achieved. Then, the proof examines these cases and establishes that none of them could result in a feasible $L(h, k)$-labeling.

We prove – by contradiction – that $\lambda_{h,k}(C) \geq 3h+2k$ if $2k \leq h < 4k$. So, assume $\lambda_{h,k}(C) < 3h+2k$. For the nomenclature, we refer to Fig. 8. Let us focus on any node a of C.

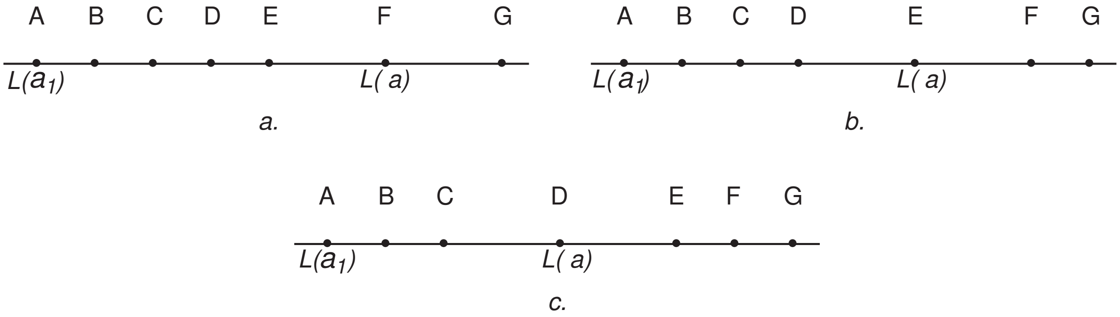


Fig. 8: Some possible relative positions of $L(a)$ and of labels of a' neighbors.

Seven cases can occur:

Case $5 - a - 1$**:** Suppose first that a has 5 neighbors whose labels are smaller than $L(a)$ and only one with label bigger than $L(a)$ (see Fig. 8.a). Hence, $L(a)$ is in position F while $L(a_1)$ is in position A.

First of all, observe that two adjacent nodes a_i and a_{i+1} cannot have their labels in consecutive positions (e.g. $L(a_i)$ in C and $L(a_{i+1})$ in D), otherwise the span would become too large, against the hypothesis $\lambda_{h,k}(C) < 3h + 2k$. In the same way, two adjacent nodes a_i and a_{i+1} cannot have their labels separated by only one label (e.g. $L(a_i)$ in C and $L(a_{i+1})$ in E) otherwise the span would be $\geq 3h+2k$. Therefore, $L(a_2)$ can be neither in position B, nor in C. Since $L(a_2)$ cannot be in G because $L(a_2) < L(a_6)$, it follows that $L(a_2)$ lies either in D or in E.

If $L(a_2)$ is in D, then $L(a_3)$ must be in G and hence $L(a_6)$ must be in E. In this way, $L(a_4)$ and $L(a_5)$ would be in B and C, in some order, achieving in any case a too large span.

Lastly, if $L(a_2)$ is in E, then $L(a_6) > L(a_2)$ must necessarily be in G and $L(a_3), L(a_4)$ and $L(a_5)$ occupy positions B, C and D in some order, leading again to a too large span.

Then – under the hypothesis $\lambda_{h,k}(C) < 3h+2k$ – this configuration never occurs.

Cases $4-a-2$**,** $2-a-4$ **and** $1-a-5$**:** If a has 4 neighbors whose labels are smaller than $L(a)$ and 2 with label bigger than $L(a)$ (see Fig. 8.b), then $L(a)$ is in E and $L(a_1$ is in A. With considerations similar to the previous ones, $L(a_2)$ can be either in D or in F or in G. $L(a_2)$ in D leads to a contradiction, as $L(a_6) > L(a_2)$ and $L(a_3)$ are in F and G in some order and, hence, $L(a_4)$ and $L(a_5)$ must lie in B and C. If $L(a_2)$ is larger than $L(a)$, then the only possibility is that $L(a_2)$ lies in F and $L(a_6)$ in G. In this case, $L(a_3), L(a_4)$ and $L(a_5)$ must be in B, C and D in sopme order, leading to a too large span. Therefore, even this case never occurs when $\lambda_{h,k}(C) < 3h+2k$.

The cases in which a has either two neighbors or one neighbor whose labels are smaller than $L(a)$ are symmetrical to the previous two cases and then omitted for the sake of brevity.

Cases $0-a-6$ **and** $6-a-0$**:** If the labels of all a's neighbors are larger (smaller) than $L(a)$, then $L(a)$ lies in A (G). These cases are both feasible in the hypothesis $\lambda_{h,k}(C) < 3h+2k$.

Case $3-a-3$**:** Finally, suppose that a has 3 neighbors whose labels are smaller than $L(a)$ and 3 with label bigger than $L(a)$ (see Fig. 8.c). In this case, $L(a_1)$ is in position A and $L(a)$ is in position D.

Since this case does not lead to any contradiction, it can occur when $\lambda_{h,k}(C) < 3h+2k$.

We have proven that only three cases can occur, i.e. the six labels of a's neighbors are: i. all smaller than $L(a)$, ii. all bigger than $L(a)$, iii. three smaller and three bigger than $L(a)$. Now we want to study which values $L(a)$ can assume and prove that no value is feasible, i.e. our hypothesis $\lambda_{h,k}(C) < 3h+2k$ is false. To this aim, we move $L(a)$ along interval $[0, 3h+2k)$ and see what happens.

$0 \leq L(a) < 2h-3k$ In this interval $L(a)$ has all six labels of a's neighbors to its right. If $L(x)$ was $\geq 2h-3k$ then the space to its right would be not sufficient to keep the span strictly smaller than $3h+2k$ and, at the same time, to fit six labels at mutual distance k and at distance $\geq h$ from $L(a)$.

$2h-3k \leq L(a) < h+2k$ $L(a)$ never lies inside this interval because there is not enough room to fit six labels to the right of $L(a)$ and not enough room to fit three labels to the left of $L(a)$. From the previous part of the proof, we know that other configurations are not possible.

$h+2k \leq L(a) < 2h$ If $L(a)$ lies in this interval, three labels are smaller than $L(a)$ and three labels are bigger than it.

$2h \leq L(a) < h+5k$ $L(a)$ never lies here, for analogous reasons with respect to the second interval.

$h+5k \leq L(a) < 3h+2k$ In this interval $L(a)$ has all six labels of a's neighbors to its left.

So, only three intervals are feasible for $L(a)$: $[0, 2h-3k)$, $[h+2k, 2h]$ and $[h+5k, 3h+2k)$. In view of the generality of a, it follows that all seven considered labels must lie in these three intervals.

The second interval is $h-2k$ wide; since $h-2k < 2k$ when $h < 4k$, we deduce that inside this interval we can fit at most two labels at mutual distance k. It follows that the other two intervals must contain at least four labels and hence they must be at least $3k$ wide each. If $2k \leq h < 3k$ this is a contradiction and the proof is concluded. If $3k \leq h < 4k$ let us consider the general $L(a)$ in the first (third) feasible interval. All six a's neighbors must have label bigger (smaller) than $L(a)$, and only two can be in the second feasible interval while four are in the third (first) one. Let us focus on the labels $L(a_i)$ and $L(a_j)$ lying inside the second feasible interval, implying that the third (first) interval is at least $h+k$ wide. If a_i and a_j are neighbors, then the second interval must be at least h wide, and this is a contradiction. If a_i and a_j have distance two in the cycle induced by a's neighbors, then consider the three nodes different from a_i, a_j and their common neighbor; they must all lie in the third (first) interval. If a_i and a_j have distance three in the cycle induced by a's neighbors, then there exist two pairs of neighbors whose labels all lie in the third (first) interval. Again, this configuration implies that the third (first) interval is at least $h+k$ wide, possible if and only if $h \geq 4k$, i.e. a contradiction. □

In the interval $2k \leq h \leq 3k$, the achieved upper and lower bounds for $\lambda_{h,k}(C)$ are not coinciding. The following result ensures us that the lower bound is not tight, at least in a subinterval:

Theorem 3.3 *If $2k < h < \frac{5}{2}k$, then $\lambda_{h,k}(C) > 3h + 2k$.*

Proof: Let be given a cellular graph C with any optimal $L(h,k)$-labeling. We consider any node a of C and study all possible values that $L(a)$ can assume, taking into account the positions of the labels of a's neighbors with respect to $L(a)$. We assume, by contradiction, that $\lambda_{h,k}(C) \leq 3h + 2k$.

Case $0 - a - 6$: Let us suppose first that all the labels of a's neighbors lie to the right of $L(a)$. It must hold that $L(a_1) \geq L(a) + h$ and that the biggest one among the labels of the a's neighbors is $\geq L(a) + h + 5k$; the same label must also be $\leq 3h + 2k$ for our hypotesis. It follows that $L(a) + h + 5k \leq 3h + 2k$, that is $L(a) \leq 2h - 3k$. Observe that the width of interval $[0, 2h - 3k]$ is strictly less than 2k when $2k < h < \frac{5}{2}k$.

Case $1 - a - 5$: If the label of one neighbor is to the right of $L(a)$, then it must be $L(a) \geq h$. On the other hand, the biggest one among the labels of the a's neighbors must be $\geq L(a) + h + 4k$ and, at the same time, $\leq 3h + 2k$. It follows $L(a) \in [h, 2h - 2k]$. The width of this interval is strictly smaller than $\frac{k}{2}$ since $h < \frac{5}{2}k$.

Cases $2 - a - 4$, $3 - a - 3$, $4 - a - 2$ and $5 - a - 1$: With reasonings identical to the previous ones, we deduce that if a has two neighbors whose label is smaller than $L(a)$ then $L(a) \in [h+k, 2h-k]$. If the neighbors whose labels are smaller than $L(a)$ are three, then $L(a) \in [h+2k, 2h]$. When four labels are smaller than $L(a)$ and two are bigger, then $L(a)$ must belong to interval $[h+3k, 2h+k]$. Lastly, when only one neighbor has its label bigger than $L(a)$ then $L(a) \in [h+4k, 2h+2k]$. The width of all these intervals is strictly less than $\frac{k}{2}$ since $h < \frac{5}{2}k$.

Case $6 - a - 0$: This case is symmetrical to the first one, and we obtain $L(a) \in [h+5k, 3h+2k]$. The width of this interval is $< 2k$ when $h < \frac{5}{2}k$.

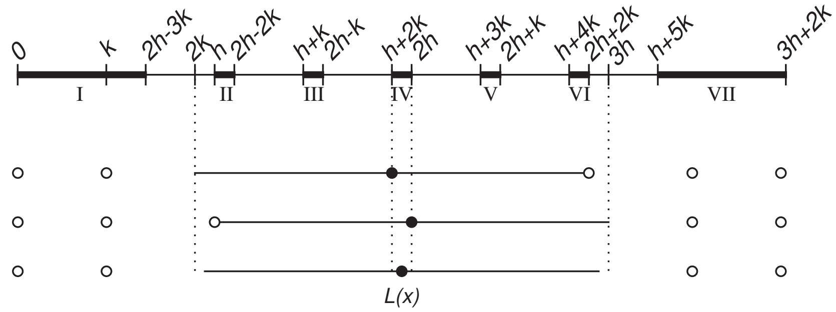


Fig. 9: The feasible intervals in the proof of Theorem 3.3.

We found seven feasible intervals for the general $L(a)$ (see Fig. 9) and we must assume that all the used labels must fall in someone of these intervals. From now on we will call *big intervals* the first one and the last one and *small intervals* all the other ones. These names derive from their widths.

Of course, for each general label $L(a)$ there are at least six labels at distance h from $L(a)$ itself, and mutually at distance k. Because of the widths of the considered intervals, we cannot fit more than two labels at mutual distance $\geq k$ inside the big intervals and no more than one label inside the small ones. So, the cardinality of the set U of used labels is no more than 9. It can also be seen that it is at least 9 because if we consider any label $L(v)$ in a small interval and any other label $L(w)$ in a neighbor small interval (e.g. $L(v)$ in the second interval and $L(w)$ in the third one), then $|L(v) - L(w)| \leq h - k < h$. So $L(v)$ eliminates two labels, that are too close, and must have other 6 labels for all v's neighbors. We can conclude that small intervals must contain one label each and that big interval must contain two labels each.

We will prove that there exists a position for $L(a)$ such that $L(a)$ cannot have other six labels at distance at least h and at mutual distance k inside the feasible intervals, proving that the hypothesis $\lambda_{h,k}(C) \leq 3h + 2k$ is a contradiction, and hence $\lambda_{h,k}(C) > 3h + 2k$. Let us focus on $L(a)$ belonging to the forth interval, i.e. to $[h + 2k, 2h]$.

If $L(a)$ lies on the left extreme of the interval, i.e. $L(a) = h + 2k$, the label $L(b)$ of each a's neighbour b must be either $\leq 2k$ or $\geq 2h + 2k$. We can fit at most two labels in the first interval. The second, third and fifth intervals are forbidden since too close to $L(a)$; we can fit at most one label in the sixth one and at most two labels in the seventh one. So, globally, we have room for at most five labels, that is not enough (see Fig. 9).

The same reasonings apply when $L(a)$ coincides with the right extreme of the interval, i.e. $L(a) = 2h$.

So, assume $L(a)$ in the open interval $(h+2k, 2h)$. Again, each $L(b)$ must be $\leq L(a)-h < 2h-h = h$ and $\geq L(a) + h > h + 2k + h = 2h + 2k$. It follows that we can fit at most four labels, two for each big interval (see Fig. 9). This concludes the proof. $\square$

On the base of the previous theorem and of the continuity of function $\lambda_{h,k}(C)$ we conjecture that $\lambda_{h,k}(C) = 4h$ if $2k \leq h \leq \frac{5}{2}k$ and $\lambda_{h,k}(C) = 2h + 5k$ if $\frac{5}{2}k \leq h \leq 3k$.

3.3 $h \geq 4k$

Theorem 3.4 *If $h \geq 4k$, then $\lambda_{h,k}(C) = 2h + 6k$.*

Proof:

Upper bound. Consider the labeled portion of cellular graph limited by bold lines in Fig. 7.b and replicate it in all directions. The definition of the labeling function is left to the interested reader.

The produced coloring is a feasible $L(h,k)$-labeling and its span is $2h+6k$.

Lower bound. Let be given a cellular graph with an optimal $L(h,k)$-labeling. Let a be any node in C. By contradiction, let us assume $\lambda_{h,k}(C) < 2h+6k$.

Cases $1-a-5$ **and** $2-a-4$**:** There exist either one or two neighbors of a whose labels are smaller than $L(a)$. With considerations very similar to those presented in the proof of Theorem 3.2 concerning adjacent nodes having their labels to the right of $L(a)$, we deduce that we always get a span $\lambda_{h,k}(C) \geq 2h+6k$; hence these cases cannot occur under the hypothesis $\lambda_{h,k}(C) < 2h+6k$.

Case $3-a-3$**:** There exist exactly three neighbors of a whose labels are smaller than $L(a)$ and three neighbors whose labels are bigger than $L(a)$.

Since $L(a_1)$ is the minimum label, if $L(a_2) < L(a)$ we achieve a too large span. It follows that $L(a_2) > L(a)$. With similar reasonings, we deduce that $L(a_1), L(a_3)$ and $L(a_5)$ lie to the left of $L(a)$ in some order, and $L(a_2), L(a_4)$ and $L(a_6)$ lie to right side.

Cases $4-a-2$ **and** $5-a-1$**:** These cases are symmetrical to cases $2-a-4$ and $1-a-5$, and hence they never occur.

Cases $0-a-6$ **and** $6-a-0$**:** These cases are both feasible in the hypothesis $\lambda_{h,k}(C) < 2h+6k$.

Up to now, we have proved that either all labels of a's neighbors lie to the same side with respect to $L(a)$ or they are three to the left and three to the right of $L(a)$. Now, let us examine which values $L(a)$ can assume.

$0 \leq L(a) < h+k$ If $L(a)$ lies in this interval, we have labels of all a's neighbors to the right of $L(a)$.

$h+k \leq L(a) < h+2k$ It is not possible to put $L(a)$ in this interval because neither case $0-a-6$ (not enough space to the right of $L(a)$) nor case $3-a-3$ (not enough space to the left of $L(a)$) can apply.

$h+2k \leq L(a) \leq h+4k$ When $L(a)$ is in this interval, we have three labels to the left of $L(a)$ and three labels to the right of $L(a)$.

$h+4k \leq L(a) \leq h+5k$ This interval cannot be used for the same reasones as the second interval.

$h+5k \leq L(a) \leq 2h+6k$ If $L(a)$ is in this interval, then all neighbors of a have labels smaller than $L(a)$, and it is feasible.

Very similarly to the proof of Theorem 3.2, it is possible to show that the assumption $\lambda_{h,k}(C) < 2h+6k$ leads to a contradiction. □

4 Hexagonal Grids

In this section we deal with the $L(h,k)$-labeling problem on hexagonal grids and we prove coinciding upper and lower bounds on $\lambda_{h,k}(H)$ for all possible values of k and $h \geq k$, except when h is in the interval $(k, 2k)$, in which case we provide sligthly different upper and lower bounds.

4.1 $k \leq h \leq 2k$

Theorem 4.1 *If $k \leq h \leq 2k$, then $2h+k \leq \lambda_{h,k}(H) \leq min\ (3h, 5k)$.*

Proof:

Lower bound. The claim directly descends from Theorem 2.1 a.

Upper bound. Consider the portion of labeled grid limited by bold lines in Fig. 10.a. We get a feasible $L(h,k)$-labeling by replicating the shown pattern of labels, and the span is $3h$. this labeling is defined by the following function:

$$L((x,y)) = ((\lceil\frac{x}{2}\rceil + \frac{3}{2}y)\bmod 4)h$$

if y is even and

$$L((x,y)) = ((\lfloor\frac{x}{2}\rfloor + 3\lceil\frac{y+1}{2}\rceil)\bmod 4)h$$

if y is odd.

The produced labeling is substantially identical to an $L(1,1)$-labeling, where all values are multiplied by h.

We can also label the hexagonal grid by an optimal $L(2,1)$-labeling, substituting each value i with ik. It is easy to see that such a coloring is feasible and its span is $5k$. It follows that $\lambda_{h,k}(H) \leq \min\ (3h, 5k)$. Combining the results for the two labelings, it follows that when $k \leq h \leq \frac{5}{3}k$ $\lambda_{h,k}(H) \leq 3h$ and when $\frac{5}{3}k \leq h \leq 2k$ $\lambda_{h,k}(H) \leq 5k$. □

4.2 $2k \leq h < 3k$

In order to make easier the reading of the proofs, in the rest of the paper we will not express anymore each label as explicit function of the coordinates of nodes in the grid, but we will refer to figures. The interested reader can easily deduce these functions from the depicted labelings.

Theorem 4.2 *If $2k \leq h < 3k$, then $\lambda_{h,k}(H) = 2h+k$.*

Proof:

Lower bound. The claim directly descends from Theorem x2.1 a.

Upper bound. Consider the upper labeled portion of hexagonal grid limited by bold lines in Fig. 10.b and constituted by three hexagons.

The $L(h,k)$-labeling is performed by replicating either this portion of labeled grid or its specular image (lower labeled portion in Fig. 10.b). It is straighforward to see that the produced $L(h,k)$-labeling is feasible and its span is $2h+k$. □

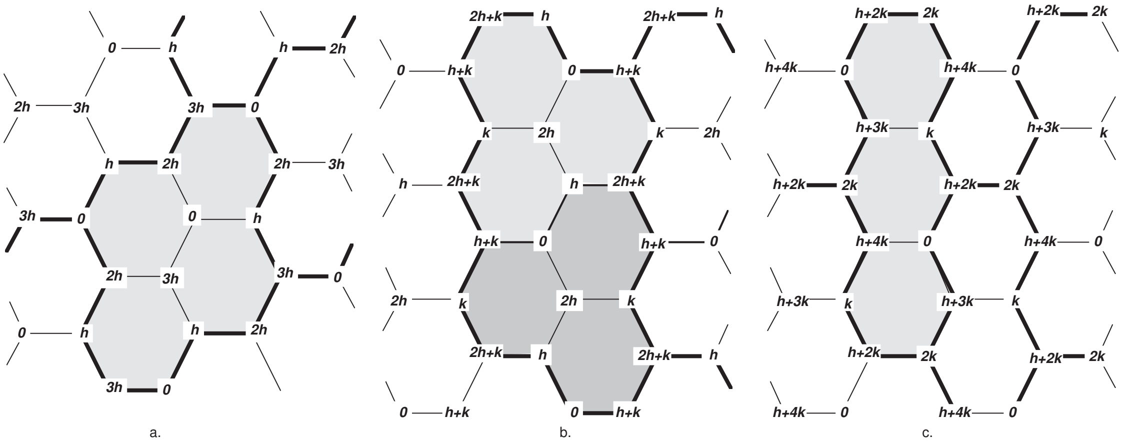


Fig. 10: Three optimally labeled portions of hexagonal grid, when a. $k \leq h \leq \frac{5}{3}k$, when b. $2k \leq h \leq 3k$ and when c. $h \geq 3k$.

4.3 $h \geq 3k$

Theorem 4.3 *If $h \geq 3k$, then $\lambda_{h,k}(H) = h + 4k$.*

Proof:

Lower bound. The claim directly descends from Theorem 2.1 b.

Upper bound. Observe that any labeling assigning to a node a label among 0 k $2k$ (respectively among $h + 2k$, $h + 3k$, $h + 4k$) and to all its neighbors labels $h + 2k$ $h + 3k$ $h + 4k$ (respectively 0, k, $2k$) is feasible and optimal. One of these labelings is shown in Fig. 10.c. □

5 Squared Grids

In this section we study the $L(h, k)$-labeling problem on squared grids. Some partial results can be found in (13), where the $L(h, k)$-labeling problem on the product of paths is studied. As a special case of this more general problem, the authors prove the following:

Theorem 5.1 *(13) If $h \geq 4h$, then $\lambda_{h,k}(S) = h + 6k$; if $3k < h < 4k$, then $\lambda_{h,k}(S) = 2h + 2k$; if $k \leq h \leq 4k$, then $2h + 2k \leq \lambda_{h,k}(S) \leq 2h + 3k - 1$.*

Here we improve the upper bound on $\lambda_{h,k}(S)$ in the interval where it is not tight.

5.1 $k \leq h \leq 2k$

Theorem 5.2 *If $k \leq h \leq 2k$, then $2h + 2k \leq \lambda_{h,k}(S) \leq min\ (4h, 2h + 3k - 1, 6k)$.*

Proof: In view of Theorem 5.1, we only have to prove the upper bounds $4h$ and $6k$.

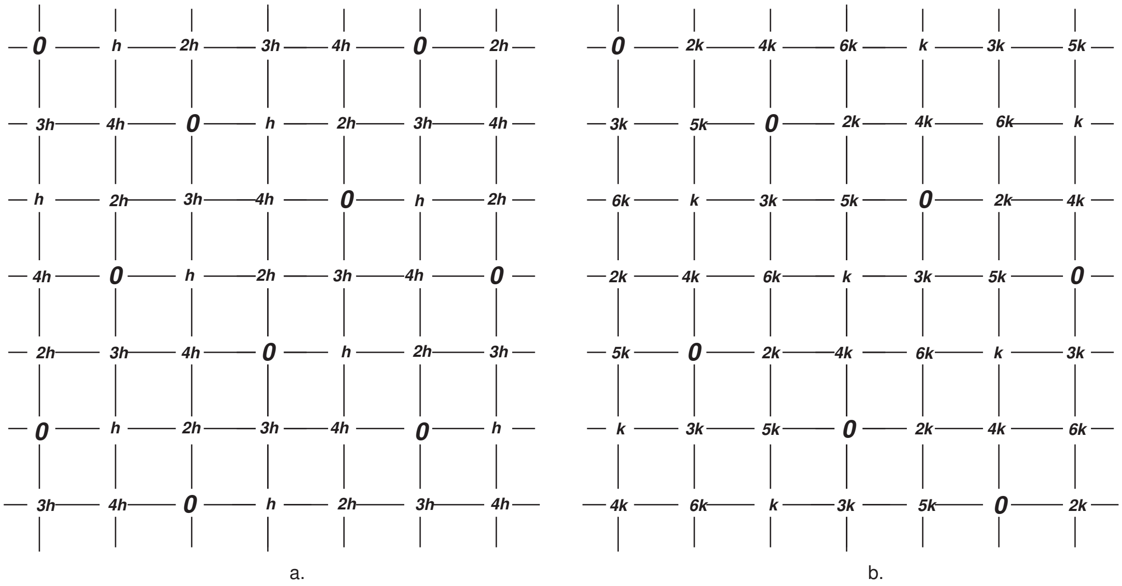


Fig. 11: Two optimally labeled portions of squared grid, when a. $k \leq h \leq \frac{3}{2}k$ and when b. $\frac{3}{2}k \leq h \leq 2k$.

We can replicate pattern $0\ h\ 2h\ 3h\ 4h$ horizontally. When we move to the next row, we shift it by two positions (see Fig. 11.a), obtaining in this way a vertical pattern $0\ 3h\ 2h\ 4h\ h$. This labeling is the same as an optimal $L(1, 1)$-labeling.

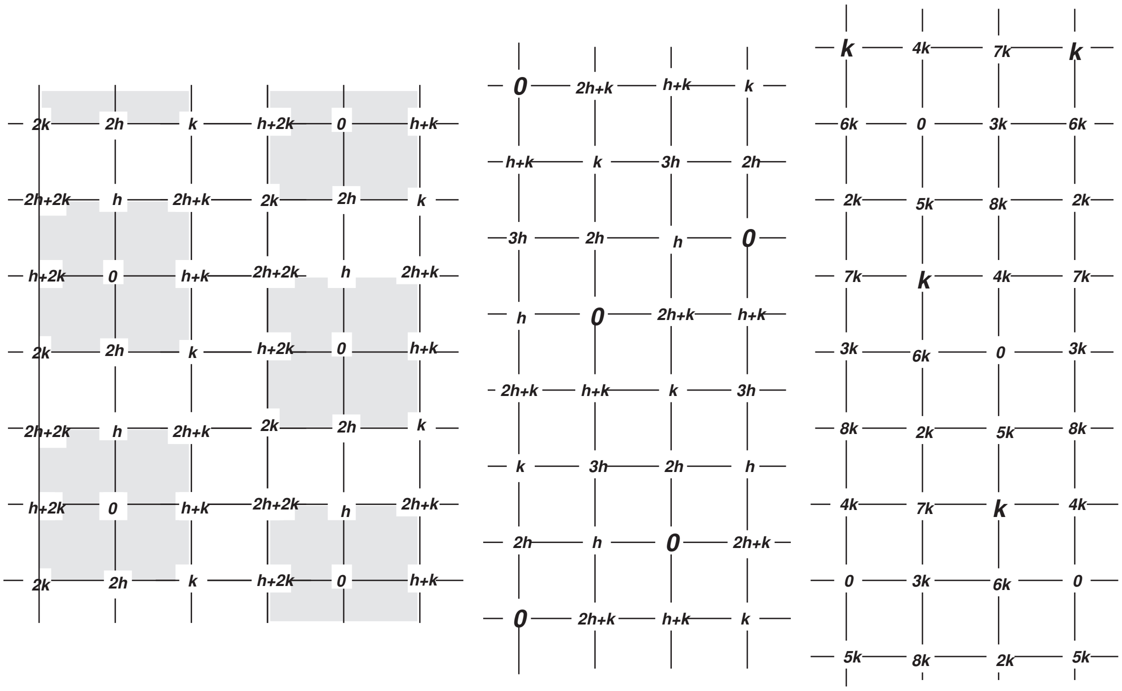
Consider now the horizontal pattern $0\ 2k\ 4k\ 6k\ k\ 3k\ 5k$. Each time we replicate it on successive rows, we shift it by two positions obtaining vertical pattern $0\ 3k\ 6k\ 2k\ 5k\ k\ 4k$ (see Fig. 11.b). Observe that this labeling can be obtained by replacing color i in an optimal $L(2, 1)$-labeling with color ik.

It is easy to see that these two $L(h, k)$-labelings are feasible and their span are $4h$ and $6k$, respectively. So, in view of these reasonings and of Theorem 5.1, we have: $\lambda_{h,k}(S) \leq \min\ (4h, 2h + 3k - 1, 6k)$. The first value is the best one when $h \leq (3k - 1)/2$, the second one is the best one when $(3k - 1)/2 \leq h \leq (3k + 1)/2$, and the third value is the best one when $h \geq (3k + 1)/2$. □

5.2 $2k \leq h < 4k$

Theorem 5.3 *If $2k \leq h \leq 3k$ then $2h + 2k \leq \lambda_{h,k}(S) \leq min(3h, 2h + 3k - 1, 8k)$.*

Proof: As in the previous proof, we only have to prove the upper bounds $3h$ and $8k$. Two $L(h, k)$-labelings can be obtained either by replicating vertical pattern $0\ h + k\ 3h\ h\ 2h + k\ k\ 2h$, as shown in Fig. 12.a, or by replicating vertical pattern $k\ 6k\ 2k\ 7k\ 3k\ 8k\ 4k\ 0\ 5k$, as shown in Fig. 12.b. Both patterns must be shifted down by three positions when changing column. Such labelings are both feasible and their spans are $3h$ and $8k$, respectively. So, combining these results with those in Theorem 5.1, we have that the bound $3h$ is better than $8k$ when $2k \leq h \leq \frac{8}{3}k$, while the second one is better than the first one when $\frac{8}{3}k \leq h \leq 3k$. Finally the bound $2h + 3k - 1$ is better than the other two in the interval $3k - 1 \leq h \leq (5k + 1)/2$, but it has positive length only when $k \leq 3$.



a. b. c.

Fig. 12: a. and b. two feasible $L(h, k)$-labelings when when $2k \leq h \leq \frac{8}{3}k$ and when $\frac{8}{3}k \leq h \leq 3k$.

Observe that the labeling in Fig. 12.b can be obtained from an optimal $L(2, 1)$-labeling by the following substitutions: $(0, 0), (1, k), (2, h), (3, h + k), (4, 2h), (5, 2h + k)$ and $(6, 3h)$. □

As for cellular graphs, also for squared grids, in the interval $2k \leq h \leq 3k$ we do not achieve tight upper and lower bounds for $\lambda_{h,k}(S)$. Nevertheless, we conjecture that $\lambda_{h,k}(S) = 3h$ if $2k \leq h \leq \frac{5}{2}k$ and $\lambda_{h,k}(C) = h + 5k$ if $\frac{5}{2}k \leq h \leq 3k$. Observe that these values guarantee the continuity of function $\lambda_{h,k}(S)$.

6 Conclusions and Open Problems

In this paper we have studied the $L(h, k)$-labeling problem on cellular, hexagonal and squared grids.

Concerning cellular and hexagonal grids, for each value of k and $h \geq k$ we have obtained exact values of the span, except in a small interval, where we provide slightly different upper and lower bounds for $\lambda_{h,k}(C)$ and $\lambda_{h,k}(H)$. Concerning the squared grid, we have improved some previously known upper bounds, reducing the gap with the lower bound.

It is easy to see that the replication schemes presented for the upper bounds lead to simple distributed algorithms to label the whole grid in constant time, provided that each node knows its coordinates in the grid.

Three open problems arise from this work.

1. The first one is to prove (or disprove) our conjectures and close the gap between upper and lower bound when $k \leq h \leq 3k$ and $\Delta = 4$ and $\Delta = 6$, and when $k \leq h \leq 2k$ and $\Delta = 3$.
2. The second one is to understand if there exists some shifting method to go from the results collected in the present paper and those presented in (20) (see Fig. 2) and vice-versa. Indeed, it is not surprising that the values of $\lambda_{h,k}$ under the 'cyclicity' assumption are bigger than ours, but it is not clear the reason why our $\lambda_{h,k}$ function is fragmented in a bigger number of segments.
3. Lastly, it would be interesting to study the $L(h,k)$-labeling problem for other (not regular) tilings, built with different shaped tiles (i.e. the edge-clique graph of the cellular graph, having degree 4, constituted by triangular and hexagonal tiles).

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