Discrete Mathematics and Theoretical Computer Science DMTCS vol. **9**, 2007, 273–292

Tribes of cubic partial cubes

Sandi Klavžar[1]† and Sergey Shpectorov[2]‡

[1] *Department of Mathematics and Computer Science, FNM, University of Maribor, Gosposvetska 84, 2000 Maribor, Slovenia*
[2] *School of Mathematics, University of Birmingham, Edgbaston, Birmingham B15 2TT, United Kingdom*

received september 16, 2005, *revised may 29, 2007*, *accepted october 15, 2007*.

Partial cubes are graphs isometrically embeddable into hypercubes. Three infinite families and a few sporadic examples of cubic partial cubes are known. The concept of a tribe is introduced as means to systematize the known examples and establish relations among them. Efficient methods of computation of tribes are developed and several concrete tribes, that include known, as well as new cubic partial cubes, are computed by hand and with the use of a computer.

Keywords: partial cube; hypercube; isometric embedding; algorithm; tribe

1 Introduction

Graphs that admit isometric embeddings into hypercubes are known as *partial cubes*. They were introduced by Graham and Pollak (22) as a model for interconnection networks and were extensively studied afterwards, see, for instance, (1; 6; 13; 15; 18; 22; 30; 36; 37) and the books (12; 25). Partial cubes also found several other applications. They are used in mathematical chemistry (27; 28), computational biology (5; 14), media theory (17), and psychology (34). The variety of partial cubes is surprisingly rich, a fact that can in particular be justified by the variety of median graphs. The latter graphs form a very special subclass of partial cubes (cf. (33)), yet it has been demonstrated in (26) that there are, roughly speaking, as many median graphs as triangle-free graphs. More precisely, it is proved that a given graph G on n vertices and m edges is triangle-free if and only if its associated graph $\widetilde{G}$ is a median graph, where $\widetilde{G}$ is a graph on $n+m+1$ vertices and $n+2m$ edges. A related result due to Bandelt and van de Vel (2) asserts that the simplex graph of an arbitrary graph is always median. Note however that the size of the simplex graph can be exponential with respect to the original graph.

It is therefore surprising that it is not easy to find regular partial cubes. This has first been encountered by Weichsel (35) who proved that among distance regular graphs only hypercubes, even cycles, and the middle-level graphs are partial cubes. Since the Cartesian product of (regular) partial cubes is again a

†Supported in part by the Ministry of Science of Slovenia under the grant P1-0297. The author is also with the Institute of Mathematics, Physics and Mechanics, Jadranska 19, 1000 Ljubljana, Slovenia.
‡Research partly supported by the Ministry of Science of Slovenia.

(regular) partial cube, there are of course several infinite series of regular partial cubes. For instance, as even cycles are partial cubes, so are the toroidal graphs $C_{2k} \,\square\, C_{2n}$, $k, n \geq 2$.

The problem of classifying cubic partial cubes is around for about a decade. As far as we know, the problem has been explicitly written down for the first time in (29), where it is proved among other statements that $P(10, 3)$ is the only partial cube among the generalized Petersen graphs. Some other examples of cubic partial cubes were known before, cf. (11), while in (3) and in (4) several new examples were constructed. It has been recently proved that among the cubic plane graphs whose faces are squares and hexagons there are precisely 5 partial cubes, all of them known before, see (9).

Clearly, even prisms $K_2 \,\square\, C_{2n}$ form an infinite series of cubic partial cubes. Besides this series, only a few sporadic examples have been known until recently[(i)], cf. Table 1. In the present paper we continue the search for cubic partial cubes and for this sake we introduce *tribes* of partial cubes. Using this concept, the related theory, and computational methods, we determine the tribes of all previously known cubic partial cubes. Among the graphs in these tribes we find 21 new cubic partial cubes. The computations were performed partly by hand and partly in the computer algebra system GAP, see (19).

We proceed as follows. In the remainder of this section basic definitions are given. Then, in Section 2, we present the list of previously known cubic partial cubes. In the subsequent section we revisit the well-known concepts of expansion and contraction and in particular introduce superconvex partitions. In Section 4 our central concept—tribe—is introduced. Then we consider the tribe of the generalized Petersen graph $P(10, 3)$ and show that it consists only of $P(10, 3)$ itself. The remaining known cubic partial cubes are planar, hence in Section 6 we restrict our investigations to the planar case and prepare the necessary theory for our computations. We also show that the even prisms $K_2 \,\square\, C_{2n}$ form a tribe. Finally, in Section 7, we presents the results of the explicit computations of the tribes of the remaining graphs from Table 1.

A subgraph H of G is called *isometric* if $d_H(u, v) = d_G(u, v)$ for all $u, v \in V(H)$, where $d_G(u, v)$ denotes the usual shortest path distance. Also, H is *convex*, if for all $u, v \in V(H)$, all shortest u, v paths from G belong to H. A convex subgraph is isometric but the converse need not be true. We call a subgraph H *superconvex* if for all $u, v \in H$, all paths between u and v of length $\leq d_G(u, v) + 1$ are contained in H. In general superconvexity is a stronger property than convexity. For bipartite graphs, however, the two properties are equivalent. A graph G is a *median graph* if there exists a unique vertex x to every triple of vertices u, v, and w such that x lies simultaneously on a shortest u, v-path, shortest u, w-path, and shortest w, v-path.

The *Cartesian product* $G \,\square\, H$ of graphs G and H is the graph with vertex set $V(G) \times V(H)$ where the vertex (a, x) is adjacent to the vertex (b, y) whenever $ab \in E(G)$ and $x = y$, or $a = b$ and $xy \in E(H)$. The Cartesian product is commutative and associative, the product of k copies of K_2 is the k-dimensional *hypercube* or *k-cube* Q_k. A graph is called *prime* (with respect to the Cartesian product) if it cannot be written as the Cartesian product of two nontrivial graphs. It has been observed in (3) that a cubic partial cube is either an even prism or it is a prime graph.

(i) In September 2005, when this paper was in preparation, David Eppstein informed us of his discovery of a connection between cubic partial cubes and simplicial arrangements of lines. This leads to two more infinite series and a number of new sporadic examples of cubic partial cubes, see (16). We briefly discuss this development in Section 8.

2 Known cubic partial cubes

In Table 1 the list of known sporadic cubic partial cubes is presented, where n stands for the number of vertices and dim for the dimension of the smallest hypercube into which the graph embeds isometrically.

n	dim	graph	ref
20	5	P(10,3)	(11; 29)
24	6	B_1	(9; 11)
30	7	B_1'	(3)
32	7	chamfered cube	(9; 11)
32	7	twisted chamfered cube	(9; 11)
36	8	B_1''	(3)
42	9	B_1'''	(3)
48	8	$\mathcal{CI}(G_1)$	(4)
48	9	$\mathcal{CI}(Q_3)$	(10; 11)
48	10	B_2	(3)
80	8	$\mathcal{CI}(G_2)$	(4)
80	8	$\mathcal{CI}(G_3)$	(4)
96	12	$\mathcal{CI}(G_4)$	(4)
96	13	$\mathcal{CI}(G_5)$	(4)
120	15	$\mathcal{CI}$(Icosahedron)	(10; 11)

Tab. 1: Known cubic partial cubes.

Let us have a closer look at the table. There we find two short series of graphs and 6 additional examples. The latter examples are the generalized Petersen graph $P(10,3)$, the truncated octahedron B_1, the chamfered cube, the twisted chamfered cube, the truncated cuboctahedron $\mathcal{CI}(Q_3)$, and the truncated icosidodecahedron $\mathcal{CI}$(Icosahedron). The truncated octahedron is also known as the permutahedron Π_3 and has appeared elsewhere, cf. (20).

The first short series $B_1, B_1', B_1'', B_1''', B_2$ with 4 new examples has been constructed in (3) by means of expansion procedures. It follows from the investigations in the present paper that this series cannot be extended.

In further search for new cubic partial cubes the following operation turned out to be useful. Let G be a graph without vertices of degree one embedded into some closed surface. Then the *cubic inflation* $\mathcal{CI}(G)$ is defined as follows, see (4). Replace each vertex $v \in V(G)$ of degree d by a $2d$-cycle Q_v, and then replace every edge uv of G by two edges joining Q_u and Q_v in such a way that an embedded cubic graph on the same surface is obtained in which all cycles Q_v are facial and all edges of G give rise to 4-faces in that embedding. $\mathcal{CI}(G)$ can also be described as the dual of the barycentric subdivision of G (4; 31) or the truncation of the medial graph of G (21) that can in turn be described as the $\{0,1,2\}$-Wythoffian (7; 8). Note that the truncated octahedron B_1 can also be represented as $\mathcal{CI}(K_4)$.

The graphs $G_1, \ldots, G_5$ that appear in Table 1 are shown in Fig. 1. The cubic inflations of these graphs provided the last series of five cubic partial cubes. We conclude this section by pointing out that it has been verified by a computer search that Table 1 is complete up to 30 vertices (3). In other words, there are just three prime cubic partial cubes on at most 30 vertices: $P(10,3)$, B_1, and B_1'.

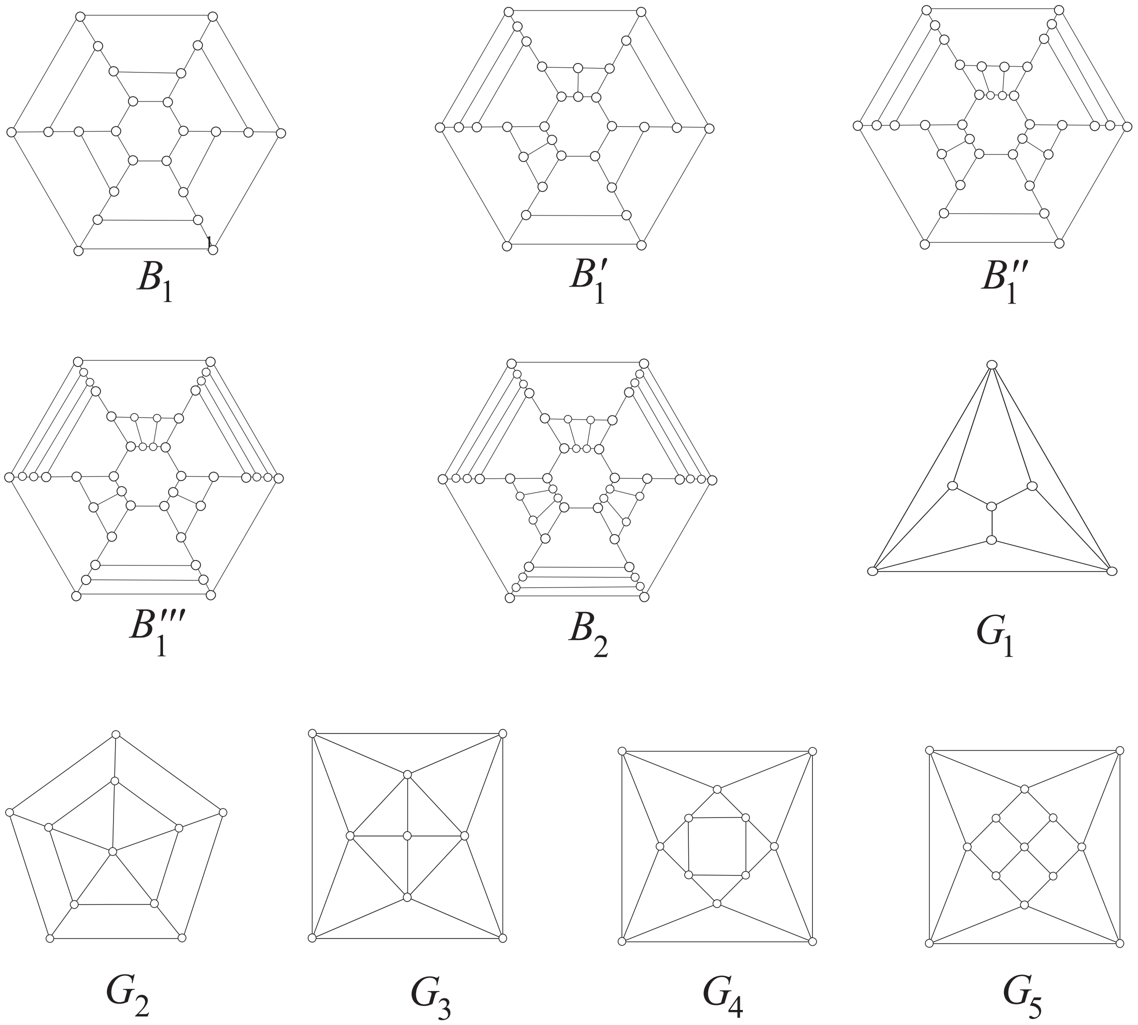

Fig. 1: Graphs from Table 1.

3 Expansion and contraction

The concept of expansion is due to Mulder (32; 33) in the context of median graphs and Chepoi (6) in the context of partial cubes (and partial Hamming graphs).

We say that two nonempty isometric subgraphs H_0 and H_1 form an *isometric cover* of a graph H provided that $H = H_0 \cup H_1$, by which we mean that $V(H) = V(H_0) \cup V(H_1)$ and $E(H) = E(H_0) \cup E(H_1)$. If H is connected then $H_0 \cap H_1 \neq \emptyset$ for every isometric cover H_0, H_1, provided that H_0 and H_1 are themselves nonempty.

Suppose H_0, H_1 is an isometric cover of H. For $i = 0, 1$, let H_i' be an isomorphic copy of H_i, and for a vertex $u \in H_0 \cap H_1$, let u_i be the corresponding vertex in H_i'. The *expansion* of H with respect to H_0, H_1 is the graph G obtained from the disjoint union of H_0' and H_1', where for each $u \in H_0 \cap H_1$ the vertices u_0 and u_1 are joined by a new edge in G. Chepoi (6) proved that a graph is a partial cube if and only if it can be obtained from K_1 by a sequence of expansions.

We next describe the operation of contraction. We say that superconvex subgraphs G_0 and G_1 form a *superconvex partition* of G if G_0 and G_1 partition G. (More precisely, $V(G_0)$ and $V(G_1)$ partition $V(G)$.) When G is partitioned by subgraphs G_0, G_1, we let $M(G_0, G_1)$ denote the set of edges of G connecting G_0 with G_1. If G is connected and G_0, G_1 are nonempty then $M(G_0, G_1)$ is also nonempty. For $e \in M(G_0, G_1)$, we let e_0 and e_1 denote the end vertices of e that are in G_0 and G_1, respectively.

Proposition 3.1 *Suppose subgraphs G_0 and G_1 partition G. Then G_0, G_1 is a superconvex partition if and only if $d_{G_0}(e_0, f_0) = d_{G_1}(e_1, f_1)$ for all $e, f \in M(G_0, G_1)$.*

Proof: Suppose G_0 and G_1 are superconvex. In particular, for $i = 0, 1$, $d_{G_i}(u, v) = d_G(u, v)$ for all $u, v \in G_i$. Suppose by way of contradiction that, say, $d_G(e_0, f_0) < d_G(e_1, f_1)$. This means that the length of a shortest path from e_1 to f_1 passing through e_0 and f_0 is $d_G(e_0, f_0) + 2 \leq d_G(e_1, f_1) + 1$. Since G_1, is superconvex, we get $e_0, f_0 \in G_1$, a contradiction. Hence, $d_G(e_0, f_0) \geq d_G(e_1, f_1)$, and we get equality by symmetry.

Now suppose $d_{G_0}(e_0, f_0) = d_{G_1}(e_1, f_1)$ for all $e, f \in M(G_0, G_1)$. Suppose by contradiction that, say, G_0 is not superconvex. Then, for some $u, v \in G_0$, there exists a path between u and v of length at most $d_G(u, v) + 1$, that is not fully contained in G_0. Let e be the first edge on this path, that crosses from G_0 to G_1, and let f be the first edge on the path, crossing back in G_0. The part of the path between e_0 and f_0 has length at most $d_G(e_0, f_0) + 1$. Furthermore, every intermediate vertex on this subpath lies in G_1. Thus, $d_G(e_0, f_0) + 1 \geq d_{G_1}(e_1, f_1) + 2$, that is, $d_G(e_0, f_0) > d_{G_1}(e_1, f_1)$. This yields a contradiction, since $d_{G_0}(e_0, f_0) \geq d_G(e_0, f_0)$. $\square$

Suppose G_0 and G_1 form a superconvex partition of G. The *contraction* of G with respect to G_0, G_1 is the graph H obtained from G by removing from it all edges $e \in M(G_0, G_1)$ and by identifying e_0 with e_1 for each e. Notice that it follows from Proposition 3.1 that $e_0 = f_0$ for $e, f \in M(G_0, G_1)$ implies $e_1 = f_1$, and vice versa. If e_i is adjacent with f_i then also e_{1-i} is adjacent with f_{1-i}. In this case, the edge $e_0 f_0$ is identified with the edge $e_1 f_1$.

The subgraphs G_0, G_1, and the set of edges $M(G_0, G_1)$ have already appeared in the literature on partial cubes. For an edge ab of a graph G let

$$
\begin{aligned}
W_{ab} &= \{u \mid u \in V(G), d_G(u, a) < d_G(u, b)\}, \\
W_{ba} &= \{u \mid u \in V(G), d_G(u, b) < d_G(u, a)\}, \\
F_{ab} &= \{uv \in E(G) \mid u \in U_{ab}, v \in U_{ba}\}.
\end{aligned}
$$

The first characterization of partial cubes is due to Djoković (13): A graph G is a partial cube if and only if it is bipartite and if for every edge ab of G the subgraphs W_{ab} and W_{ba} are (super)convex.

Proposition 3.2 *If G_0, G_1 is a superconvex partition of a graph G then $G_0 = W_{e_0e_1}$, $G_1 = W_{e_1e_0}$, and $M(G_0, G_1) = F_{e_0e_1}$ for every $e \in M(G_0, G_1)$.*

Proof: Set $a = e_0$ and $b = e_1$. It suffices to show that $G_0 \subseteq W_{ab}$ and, symmetrically, $G_1 \subseteq W_{ba}$. Suppose $d_G(u, b) \leq d_G(u, a)$ for some $u \in G_0$. Then there exists a path from a to u via b, that has length $d_G(u, b) + 1 \leq d_G(u, a) + 1$. By superconvexity of G_0, we get that $b \in G_0$, a contradiction. □

We end this section with a remark that if a graph G is obtained from a graph H via expansion with respect to an isometric cover H_0, H_1 then the images of H_0 and H_1 in G (that is, H_0' and H_1') form a superconvex partition of G. Consequently, H can be recovered from G via the corresponding contraction. Reversely, if a graph H is obtained from a graph G via contraction with respect to a superconvex partition G_0, G_1 then the images of G_0 and G_1 in H form an isometric cover, and G can be recovered from H via the expansion with respect to this isometric cover. In this sense, expansion and contraction are inverse to each other.

4 Tribes

The graphs, on which we now focus our attention, are regular partial cubes, especially, cubic partial cubes. The class of partial cubes is closed with respect to expansions and contractions. This is, in essence, the result of (6). Let us now investigate conditions, under which expansion and contraction preserve regularity. The following two propositions are applicable to arbitrary regular graphs (not necessarily partial cubes).

Suppose H is a regular graph of degree n. Suppose further that G is an expansion of H with respect to an isometric cover H_0, H_1. For $v \in H_0 \cap H_1$ let a_v be the number of neighbors of v outside of H_0, b_v the number of neighbors outside of H_1, and c_v the number of neighbors in $H_0 \cap H_1$.

Proposition 4.1 *Under the above assumptions, G is regular if and only if one of the following is true:*

(1) *$H_0 = H_1 = H$, and $G = H \,\square\, K_2$ is of degree $n + 1$; or*

(2) *G has degree n and, furthermore, $a_v = b_v = 1$ and $c_v = n - 2$ for all $v \in H_0 \cap H_1$.*

Proof: If $H_0 = H_1 = H$ then, clearly, (1) holds. Otherwise, there is a vertex outside of $H_0 \cap H_1$. The corresponding vertex of G has degree n. Therefore, if G is regular, it must be of degree n.

Let $v \in H_0 \cap H_1$. Clearly, $a_v + b_v + c_v = n$. Since the degree of v_0 in G is $1 + b_v + c_v$ and the degree of v_1 is $a_v + 1 + c_v$, we get the claim. □

In the cubic case, G is again cubic if and only if $a_v = b_v = c_v = 1$. In particular, $H_0 \cap H_1$ must be a disjoint union of edges. We will call such an expansion a *cubic expansion*.

Suppose now that H is obtained from G by contraction with respect to a superconvex partition G_0, G_1. For $i = 0, 1$, set $U_i = \{e_i \mid e \in M(G_0, G_1)\}$. According to Proposition 3.1, the mapping sending e_0 to e_1 for all $e \in M(G_0, G_1)$, establishes an isomorphism between distance spaces U_0 and U_1. In particular, this mapping is also an isomorphism of the graphs induced on U_0 and U_1.

Suppose G is regular of degree n.

Proposition 4.2 *The contraction H is regular if and only if one of the following holds:*

(1) $U_0 = G_0$ *and* $U_1 = G_1$*; in this case* H *has degree* $n - 1$*;*

(2) *the graph on* U_0 *is regular of degree* $n - 2$*; here* H *has degree* n*.*

Proof: If $U_0 = G_0$ and $U_1 = G_1$ then, clearly, (1) holds. If there is a vertex in G outside of $U_0 \cup U_1$ then the corresponding vertex of H has degree n. So H must be of degree n. For $e \in M(G_0, G_1)$, let a be the number of neighbors of e_0 in $G_0 \setminus U_0$ and b the number of neighbors in U_0. Then $n = a + b + 1$. Since e_1 has the same number b of neighbors in U_1, it follows that e_1 must have exactly a neighbors in $G_1 \setminus U_1$. Therefore, the unified e_0, e_1 vertex of H has degree $a + b + a$. This yields $a = 1$ and $b = n - 2$. □

When G is cubic, the contraction H is also cubic if and only if the graph on U_0 is a disjoint union of edges. In this case we call the contraction a *cubic contraction*.

We can now introduce our central concept—tribes. If G is a cubic partial cube then every graph obtained from G by cubic expansion or cubic contraction is again a cubic partial cube. A *tribe* of cubic partial cubes is a minimal nonempty collection of such graphs, closed under cubic expansions and cubic contractions. The tribe containing a particular graph G will be referred to as the *G-tribe*.

5 The case of $P(10, 3)$

In this section we prove the following

Proposition 5.1 *The generalized Petersen graph* $P(10, 3)$ *admits no cubic expansions and no cubic contractions. Consequently, the* $P(10, 3)$*-tribe consists only of* $P(10, 3)$ *itself.*

We will prove this proposition in a sequence of lemmas, that also introduce some of the ideas involved in tribe computations.

Lemma 5.2 *If the contraction of a cubic graph* G *with respect to a superconvex partition* G_0, G_1 *is a cubic contraction then every edge from* $M(G_0, G_1)$ *is contained in an isometric* 4*-cycle. In particular,* $P(10, 3)$ *admits no cubic contractions.*

Proof: Suppose $e \in M(G_0, G_1)$. By the comment after Proposition 4.2, the graph $U_0 = \{f_0 | f \in M(G_0, G_1)\}$ has degree one. If $f \in M(G_0, G_1)$ is such that f_0 is adjacent to e_0 then e_1, e_0, f_0, f_1 induce an isometric 4-cycle, as follows from Proposition 3.1.

Since the girth of $P(10, 3)$ is six, this graph admits no cubic contractions. □

For a cycle C of a cubic graph H, an edge e of H is *adjacent* to C at vertex $v \in C$ if e is incident to v and it does not lie on C. Recall that by the remark after Proposition 4.1, if an isometric cover H_0, H_1 defines a cubic expansion then $H_0 \cap H_1$ is a disjoint union of edges.

Lemma 5.3 *Suppose an isometric cover* H_0, H_1 *of a cubic graph* H *defines a cubic expansion of* H*. Suppose* C *is a cycle of* H *that is convex as a subgraph of* H*. Suppose further that* e *is an edge that is adjacent to* C *at a vertex* v*. If* e *is contained in* $H_0 \cap H_1$ *then the following hold:*

(1) C *has even length and* $C \cap (H_0 \cap H_1) = \{v, u\}$*, where* u *is the vertex opposite* v *on* C*;*

(2) *one of the two halves of* C *between* v *and* u *belongs to* H_0 *and the other one belongs to* H_1*;*

(3) *the edge adjacent to* C *at* u *lies in* $H_0 \cap H_1$*.*

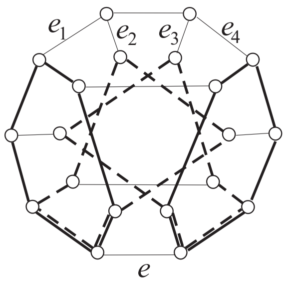

Fig. 2: Relation $\sim$ in $P(10,3)$.

Proof: Let a and b be the neighbors of v on the cycle C. Since e is contained in $H_0 \cap H_1$, we have that both a and b do not lie in $H_0 \cap H_1$. Without loss of generality, $a \in H_0 \setminus (H_0 \cap H_1)$ and $b \in H_1 \setminus (H_0 \cap H_1)$. Considering the path from a to b around C (avoiding v), we see that C contains another vertex u that is contained in $H_0 \cap H_1$. Since H_0 is isometric there is a shortest path between u and v that is fully contained in H_0. Since C is convex, this path is contained in C. Similarly, there is a shortest path between u and v that is contained in $H_1 \cap C$. Since $a \notin H_1$ and $b \notin H_0$, C contains two different shortest paths between u and v. This means that C has even length and that u is opposite v on C.

This also means that such a vertex u is unique. Therefore, the two edges on u, that lie on C, are not contained in $H_0 \cap H_1$. It follows that the edge that is adjacent to C at u is in $H_0 \cap H_1$. □

When edges e and f of a graph H are adjacent at opposite vertices to a convex cycle of even length, we will write $e \sim f$. This relation extends by reflexivity and transitivity to an equivalence relation on $E(H)$. This equivalence relation will be denoted by the same symbol $\sim$. It follows from Lemma 5.3 that if H_0, H_1 defines a cubic expansion of H and e lies in $H_0 \cap H_1$ then the entire $\sim$-equivalence class of e is contained in $H_0 \cap H_1$.

Let us now see, what this gives for $P(10,3)$. Every 6-cycle in $P(10,3)$ is convex. Every edge e is adjacent to four 6-cycles, cf. Fig. 2, where the four 6-cycles are emphasized.

Each of the 6-cycles leads to a new edge in the equivalence class of e, on Fig. 2 these are the edges e_1, e_2, e_3, and e_4. Continuing by transitivity, one establishes that all edges of $P(10,3)$ form a single $\sim$-equivalence class. This means that no cubic expansion is possible for $P(10,3)$. Combining this fact with Lemma 5.2 completes the proof of Proposition 5.1.

6 Plane graphs

The fourteen remaining examples from Table 1 are plane graphs with convex faces. (We say that a face of a plane graph is convex (respectively, isometric) if the boundary cycle of the face is a convex (respectively, isometric) subgraph.) These properties lead to a very explicit description of possible isometric covers and superconvex partitions and, as a result, to efficient algorithms for computing all cubic expansions and cubic contractions. We realized these algorithms in GAP (19) and used them for the tribe computations presented in the next section.

We also prove in this section that a graph, obtained by cubic expansion from a cubic plane graph with convex faces, inherits these same properties, so that our algorithms are again applicable. Cubic contractions preserve planarity, but not necessarily convexity of the faces. However, as we show in this

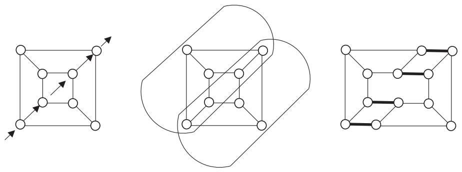

Fig. 3: A straight-ahead walk in Q_3, the corresponding isometric cover and cubic expansion.

section, non-convex faces can only arise in a very special situation that never occurs for our graphs.

Let H be a plane cubic partial cube and ab an edge of H. Let W be the edge-face walk on H that starts from ab and continues as follows. Going along an edge uv from u to v, we enter the face at v that is not incident with u. (The edge uv is adjacent to this face at v.) In this face we proceed to the opposite vertex v' and exit the face along the edge adjacent to the face at v'. We call W a *straight-ahead walk*; see Fig. 3 where a straight-ahead walk in Q_3 is shown.

Proposition 6.1 *Suppose H is a plane cubic graph with convex faces. If H_0, H_1 form an isometric cover of H, defining a cubic expansion, then there exists a straight-ahead walk W such that*

(1) *W has no self-intersections;*

(2) *for $i = 0, 1$, H_i is induced by all vertices on one side of W and on W itself;*

(3) *$H_0 \cap H_1$ is the disjoint union of the edges lying on W.*

Suppose H and H_0, H_1 are as above. For the proof of Proposition 6.1 we need the following simple fact to be used later as well and an important lemma classifying possible intersections of faces with $H_0 \cap H_1$.

Proposition 6.2 *For a graph H and subgraphs H_0 and H_1, suppose $H = H_0 \cup H_1$. Then H_0, H_1 is an isometric cover if and only if $d_{H_0}(u, v) = d_{H_1}(u, v)$ for all $u, v \in H_0 \cap H_1$.*

Lemma 6.3 *Suppose F is a face of H. Then one of the following holds:*

(1) *F fully lies in H_i for some $i = 0, 1$, and meets $H_0 \cap H_1$ trivially;*

(2) *F fully lies in H_i for some $i = 0, 1$, and meets $H_0 \cap H_1$ in an edge; or*

(3) *F meets $H_0 \cap H_1$ in two opposite vertices v and u, the two edges adjacent to F at v and u are contained in $H_0 \cap H_1$, and the two halves of F between v and u are contained in different H_i.*

Proof: If F meets $H_0 \cap H_1$ trivially then, clearly, (1) holds. Suppose $v \in F \cap (H_0 \cap H_1)$ and let e be the unique edge on F that is contained in $H_0 \cap H_1$. If e is adjacent to F then (3) holds by Lemma 5.3, since F is convex. It remains to consider the case where e lies on F. We claim that in this case (2) holds. Suppose by way of contradiction that there is a third vertex w in $F \cap (H_0 \cap H_1)$. Let f be the edge on w that is contained in $H_0 \cap H_1$. If f is adjacent to F then (3) must holds, a contradiction. Hence, f lies on F. Taking, if necessary, the other end of f as w, we can assume that w is not opposite v. Since F is convex and v and w are not opposite, there exists a unique shortest path between v and w. However, by Proposition 6.2, this shortest path should be contained in both H_0 and H_1, a contradiction, since $H_0 \cap H_1$ is a disjoint union of edges. □

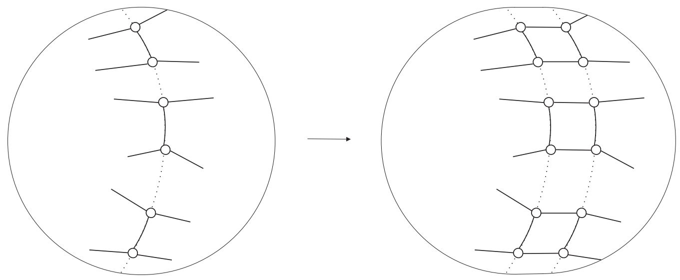

Fig. 4: Cubic expansion.

We are now ready to prove Proposition 6.1.

Proof of Proposition 6.1. Suppose e is an edge in $H_0 \cap H_1$ and let W be the corresponding straight-ahead walk. Since the faces of H are convex, Lemma 5.3 implies that all edges on W are contained in $H_0 \cap H_1$. This shows that $H_0 \cap H_1$ is a disjoint union of several straight-ahead walks. Suppose W intersects another walk W' contained in $H_0 \cap H_1$. (We allow for the equality $W' = W$.) There are two possibilities for the intersection. If the intersection is at a vertex w then each walk has an edge on w (and this cannot be the same edge). This yields a contradiction, since the edges in $H_0 \cap H_1$ are disjoint. If the intersection of W and W' is at a face F then Lemma 6.3 (3) gives a contradiction. Therefore, $H_0 \cap H_1$ has no self-intersections.

If $H_0 \cap H_1$ consists of more than one walk then cutting the sphere through all these walks splits the sphere into more than two parts. However, this means that H_0 and H_1 cannot be both connected. (Notice that this topological argument also can be used to show the absence of self-intersections.) Thus, $H_0 \cap H_1 = W$ is a single, non self-intersecting walk. This proves (1) and (3).

Considering the faces fully lying on one side of W, we see that either (1), or (2) of Lemma 6.3 must hold for each of these faces. Now connectivity implies (2). $\square$

Proposition 6.4 *If G is obtained by cubic expansion from a plane cubic graph with convex faces then G is also a plane graph with convex faces.*

Proof: Topologically, in this situation the cubic expansion amounts to cutting the plane graph H along the walk $W = H_0 \cap H_1$ and inserting a band with the new edges drawn across it, see Fig. 4.

So G has a natural structure of a plane graph. With respect to this structure, G has three kinds of faces: (a) the faces of H, that are not affected by the expansion; (b) the faces affected by the expansion; these are the edges as in Lemma 6.3 (3); the gonality of these edges in G increases by two; and (c) the new 4-gonal faces corresponding to the edges on W. Notice that the reverse operation of contraction transforms every path γ' on G into a path γ on H. The length of γ is the length of γ' minus the number of times γ' crosses the inserted band.

Suppose F' is a face of G and let F be the corresponding face of H. (If F' is of type (c) then F is an edge. Notice that an edge is convex as a subgraph of H.) Let v' and u' be vertices of F' and let v and u be the corresponding vertices of F. Consider a shortest path γ' in G from v' to u' and let γ be the corresponding path in H from v to u. Suppose first that v' and u' are on the same side of the band (that is, in the same H_i' for some $i = 0, 1$). If γ' crosses the band, then it crosses the band at least twice, so $d_G(v', u') \geq d_H(v, u) + 2$. However, $d_H(v, u) = d_{H_i}(v, u) = d_{H_i'}(v', u')$, and so $d_G(v', u') > d_{H_i'}(v', u')$. This is a contradiction, since H_i' is a subgraph of G. Thus, γ' is fully contained in H_i'. Now, since H_i' is a copy of H_i, it follows that γ is a shortest path from v to u and so it is contained in F, since F is convex. Hence also γ' is contained in F'.

When v' and u' lie on different sides of the band, we similarly consider a shortest path γ' in G and the corresponding γ in H. In this case we show that the length of γ is one less than the length of γ' and that again γ is a shortest path from v to u. It follows that γ lies on F and, consequently, γ' lies on F'. □

Next we turn to the operation of contraction. We start with a rather general lemma.

Lemma 6.5 *Suppose G_0, G_1 form a superconvex partition of a graph G and C is an isometric cycle in G. Then one of the following holds:*

(1) *C is fully contained in G_i for some $i = 0, 1$; or*

(2) *C has even length and it meets $M(G_0, G_1)$ in two opposite edges.*

Proof: If none of the edges on C is in $M(G_0, G_1)$ then case (1) holds. Suppose an edge e from C belongs to $M(G_0, G_1)$. Clearly, after crossing from one part of the partition to the other via e, C must return back, so C contains a second edge, say f, from $M(G_0, G_1)$. Since both parts G_i are convex, e_1 and f_1 cannot lie on a shortest path between e_0 and f_0 and, vice versa, e_0 and f_0 cannot lie on a shortest path between e_1 and f_1. Since C is isometric, this means that the part of C between e_0 and f_0 is a shortest path between these two vertices and, similarly, the part of C between e_1 and f_1 is a shortest path between them. Finally, Proposition 3.1 forces C to have even length and e and f to be opposite. □

We now return to the plane graphs. Suppose G is such a graph and suppose that all faces of G are isometric. Let G_0, G_1 form a superconvex partition of G. Since the faces are isometric, the above lemma implies that all faces, containing edges from $M(G_0, G_1)$, have even gonality and meet $M(G_0, G_1)$ in a pair of opposite edges. Let us make the edges from $M(G_0, G_1)$ the vertices of a new graph, in which two edges are adjacent whenever they are opposite each other on a face. Clearly, this is a graph of degree two and hence it is a union of cycles. Each of these cycles is called a *zone* (or, sometimes, a *railroad*). With each zone one can associate a cyclic *zone walk* on the sphere, in which G is imbedded. Namely, starting from the middle of an edge we go across the face on one side of the edge to the middle of the opposite edge (which is again in the zone), cross into the next face, and so on until we return to the middle of the initial edge from the other side.

Proposition 6.6 *Suppose G is a plane graph with isometric faces and G_0, G_1 form a superconvex partition of G. Then $M(G_0, G_1)$ is a single zone. Furthermore, the corresponding zone walk has no self-intersections and each of G_0 and G_1 consists of all vertices on one side of the walk.*

Proof: If a zone in $M(G_0, G_1)$ had a self-intersection or if there were two zones or more, then removing from G all edges from $M(G_0, G_1)$ would result in more than two connected components, which is not the case. So $M(G_0, G_1)$ is a single zone and the corresponding walk has no self-intersections. The last claim also follows, since G_0, G_1 are the connected components of G when all edges from $M(G_0, G_1)$ are removed. □

Following the zone walk corresponding to the zone $M(G_0, G_1)$, let us write down the gonalities (the number of edges) of the faces that we cross. This gives a cyclic sequence $g_1, g_2, \ldots, g_m$, where m is the number of edges/faces in the zone. We will call this sequence the *zone sequence*.

Proposition 6.7 *Suppose a plane graph G is cubic with convex faces. Then the contraction defined by G_0, G_1 is a cubic contraction if and only if 4 appears in the zone sequence in every other place.*

Proof: First of all, we argue that every 4-cycle in G is a face. Indeed, suppose C is a 4-cycle. Let F be a face that contains two consecutive edges from C. Since F is convex, we must have $F = C$.

Now for $e, f \in M(G_0, G_1)$, we have that e_0 and f_0 are adjacent if and only if e_0, f_0, f_1, e_1 is a 4-cycle (hence a 4-gonal face). So the claim follows from the remark after Proposition 4.2. □

Finally, we show that contractions preserve planarity and discuss under which conditions it preserves convexity of faces.

Proposition 6.8 *If G is a plane graph with isometric faces then the graph H resulting from the contraction with respect to G_0, G_1 is again a plane graph with isometric faces.*

Proof: By Proposition 6.6, $M(G_0, G_1)$ is a single zone and the corresponding walk has no self-intersections. This implies that the contraction amounts to cutting out a band from the sphere (cf. Fig. 4 with the reversed arrow) and gluing the two disks back together. So H has a natural plane structure. Furthermore, the faces of G that contain no edges from $M(G_0, G_1)$ retain their gonality in H, while the gonality of the faces containing edges from $M(G_0, G_1)$ goes down by two. The 4-gonal faces of this latter type disappear altogether.

Suppose F' is a face of H and F is the face of G from which F' is produced. Suppose v' and u' are vertices of F', and suppose γ' is a shortest path between them. Let G_0' and G_1' be the images in H of G_0 and G_1, respectively. Recall that each G_i' is isomorphic to G_i, and that G_0', G_1' form an isometric cover.

Suppose first that v' and u' are contained together in some G_i'. Since G_i' is isometric in H, there is a shortest path between v' and u' that is fully contained in G_i'. Let v and u be the preimages of v' and u' in G_i. Clearly, $v, u \in F$. We have $d_H(v', u') = d_{G_i'}(v', u') = d_{G_i}(v, u) = d_G(v, u)$. Since F is isometric, $d_G(v, u) = d_F(v, u)$. Also, $d_{F'}(v', u') \le d_F(v, u)$. This implies $d_{F'}(v', u') \le d_H(v', u')$, and since F' is a subgraph of H, the equality must hold.

Suppose now that v' and u' are in different G_i''s. In particular, neither of the two vertices is in $G_0' \cap G_1'$. Without loss of generality we may assume that $v' \in G_0'$ and $u' \in G_1'$. Let v and u be the preimages of v' and u', respectively. We have that $v, u \in F$. Since every path from v to u necessarily contains an edge from $M(G_0, G_1)$, we must have $d_H(v', u') \le d_G(v, u) - 1$. To prove the reverse inequality, consider a shortest path γ' from v' to u'. Let $x_1', x_2', \ldots, x_k'$ be all the vertices from this path that are contained in $G_0' \cap G_1'$. The vertices on γ' that precede x_1' are in G_0'. The vertices that follow x_k' are contained in G_1'. Also, in view of Proposition 6.2, for every $i = 1, \ldots, k-1$, we can substitute the subpath from x_i' to x_{i+1}' by a subpath of equal length, that is fully contained in G_0'. Thus, without loss of generality, we may assume that all vertices preceding x_k' are in G_0'. Now we can lift γ' to a path γ in G as follows: We lift the part of γ' from v' to x_k' into G_0, we lift the tail of γ', from x_k' to u', into G_1, and we connect the two parts of γ by the edge from $M(G_0, G_1)$ between the two preimages of x_k'. Clearly, γ is a path from v to u and its length is one plus the length of γ'. This shows that $d_G(v, u) \le d_H(v', u') + 1$. Thus, $d_G(v, u) = d_H(v', u') + 1$.

Finally, since $d_F(v, u) = d_{F'}(v', u') + 1$ and since $d_F(v, u) = d_G(v, u)$, we get $d_{F'}(v', u') = d_H(v', u')$. □

Unfortunately, it is not true that F' is convex whenever F is. However, the conditions under which F' is not convex are rather restrictive and they do not occur in practice very often.

Suppose H is obtained from G via a cubic contraction with respect to G_0, G_1. For $i = 0, 1$, let G'_i denote the image of G_i in H.

Proposition 6.9 *Under the above assumptions, let F' be a face of H and F be the corresponding face of G. Suppose further that F is convex. Then F' is not convex if and only if it contains two or more edges from $G'_0 \cap G'_1$.*

Proof: Suppose the edges e' and f' from $G'_0 \cap G'_1$ lie on F'. Pick v' incident to e' and u' incident to f' so that $m = d_H(v', u')$ is minimal. Since G'_0 is an isometric subgraph of H, there is a shortest path α' from v' to u' that is fully contained in G'_0. Similarly, there is a shortest path β' between v' and u' that is fully contained in G'_1. Let a' (respectively, b') denote the vertex that follows v' on α' (respectively, β'). By the minimality of m, neither a', nor b' is incident with e'. Since e' lies on F', we conclude that either a', or b' does not lie on F'. Hence F' is not convex.

Now suppose F' is not convex, namely, there is a shortest path γ' between $v', u' \in F'$, that is not contained in F'. We will assume that the length $m = d_H(v', u')$ of γ' is minimal among all such choices of v', u', and γ'. In particular, none of the intermediate vertices from γ' lies on F'. Note that this property holds for every path between v' and u' that has length m and is not fully contained in F'. Let $v, u \in F$ be preimages of v' and u'. Let $i, j \in \{0, 1\}$ be defined by $v \in G_i$ and $u \in G_j$.

Suppose some intermediate vertex t' from γ' is contained in $G'_0 \cap G'_1$. Since G'_0 and G'_1 are isometric in H, we can assume without loss of generality that the part of γ' from v' to t' is fully contained in G'_i and the part from t' to u' is fully contained in G'_j. Such a path can be lifted to a shortest path γ from v to u. (The length of γ is m if $i = j$, and it is $m + 1$ if $i \neq j$.) Since F is convex, we must have that γ lies on F, and therefore, γ' lies on F', a contradiction. Thus, no intermediate vertex of γ' is in $G'_0 \cap G'_1$. This means that all intermediate vertices are contained in $G'_k \setminus (G'_0 \cap G'_1)$ for some $k = 0, 1$.

If $i = k$ or $j = k$ then γ' can be lifted to a shortest path γ from v to u, again yielding a contradiction with convexity of F. Hence $i = j = 1 - k$. In particular, this implies that $v', u' \in G'_0 \cap G'_1$.

If v' and u' belong in the same connected component of $G'_0 \cap G'_1$ (which is an edge!), then $m = 1$, clearly giving a contradiction. Thus, v' and u' are incident to two different edges, say e' and f', from $G'_0 \cap G'_1$. It remains to show that both e' and f' lie on F'. However, this is clear. Indeed, by the minimality of m, the first edge along γ' does not lie on F'. Hence the other two edges on v' must lie on F'; one of them is e'. By symmetry, also f' lies on F'. □

The condition for the non-convexity of F' can also be stated in terms of G, F and the zone $Z = M(G_0, G_1)$. Namely, F' is non-convex if and only if the zone walk corresponding to Z crosses two different quadrangular faces adjacent to F (but not F itself).

We conclude this section with the following illustration of our theory.

Proposition 6.10 *The even prisms $K_2 \,\square\, C_{2n}$ form a tribe.*

Proof: It suffices to show that every cubic expansion of $K_2 \,\square\, C_{2n}$ coincides with $K_2 \,\square\, C_{2n+2}$ and that every cubic contraction of $K_2 \,\square\, C_{2n}$ coincides with $K_2 \,\square\, C_{2n-2}$.

We start with contractions. First of all, the cube $Q_3 = K_2 \,\square\, C_4$ has no cubic contractions, since it has no non-quadrangular faces. Let $n \geq 3$. In this case the prism has, up to symmetries, two types of edges: (1) the edges of the two $2n$-gonal faces (top and bottom of the prism); and (2) the side edges, each of which is incident with two quadrangular faces (sides of the prism). The edges of the second kind form a zone. This zone does not lead to a cubic contraction because the zone sequence is the constant 4. Each of the remaining zones looks the same: the zone walk goes across the top, then down the side, then across the bottom in the opposite direction, and then back up the opposite side. Manifestly, each of these zones defines a superconvex partition and the resulting graph is $K_2 \,\square\, C_{2n-2}$.

We now turn to expansions. When $n = 2$, we have the cube Q_3, which has only one class of edges. Hence all straight-ahead walks are as in Fig. 3. Each of them defines an isometric cover and the resulting graph is the hexagonal prism. Now suppose $n \geq 3$. As above, we distinguish two kinds of edges. If a straight-ahead walk contains a side edge, then this walk goes up this edge, continues across the top, goes down the opposite side edge, and then crosses the bottom face back to the origin. Such straight-ahead walks do define isometric covers; indeed, the corresponding cubic expansions are the reverse operations to the cubic contractions from the preceding paragraph.

It remains to consider the straight-ahead walks containing the edges of the $2n$-gonal top and bottom faces. Suppose, say, e_1 is an edge of the top face. The straight-ahead walk starting from e_1 goes diagonally down the next side face, then along an edge f_1 of the bottom face, climbs diagonally up the next side face, goes along the top edge e_2, and so on. If n is odd, then the walk goes around the prism and eventually self-intersects. Suppose n is even (and hence $n \geq 4$). In this case the straight-ahead walk has no self-intersections. However, it still does not define an isometric cover. Indeed, let a be the second vertex on e_1 and let b be the first vertex on e_2. (We count in the direction of the walk.) Let H_0, H_1 be the cover defined by the walk, where H_0 contains the top face and H_1 contains the bottom. Manifestly, $d_{H_0}(a, b) = 3$, while $d_{H_1}(a, b) = 5$. Thus, H_0, H_1 is not an isometric cover. □

7 Explicit tribes of partial cubes

The theory that we have developed in the previous section leads to efficient algorithms listing all cubic expansions and contractions of a given plane cubic partial cube with convex faces. Suppose H and G are such graphs. To construct all cubic expansions of H, do the following:

(1) Construct all straight-ahead walks in H. Notice that each edge lies on a unique straight-ahead walk, so the straight-ahead walks partition the edge set of H.
(2) Select those straight-ahead walks W that have no self-intersections.
(3) For each such W, construct H_0 and H_1 as subgraphs induces by the vertices on one side of W and on W.
(4) Select isometric covers H_0, H_1 by verifying the distance condition as in Proposition 6.2. Notice that $H_0 \cap H_1 = W$.

To determine all cubic contractions of G, do the following:

(1) Construct all zones in G. Again, the zones partition the edge set of G, and so each edge needs to be dealt with only once.
(2) Select all zones Z, whose zone walks have no self-intersections and such that 4 appears in the zone sequence in every other place.
(3) For each such Z, construct G_0 and G_1 as the connected components of G with all edges from Z removed.
(4) Select superconvex partitions G_0, G_1 by verifying the distance condition as in Proposition 3.1.

In the case of cubic expansions, planarity of the resulting graph and convexity of its faces is assured by Proposition 6.4. Thus, the above algorithms can immediately be applied to construct further graphs from the same tribe. In the case of cubic contractions, planarity of the resulting graph is assured by Proposition

6.8, while convexity of faces must be verified using Proposition 6.9 before the computation of the tribe can proceed.

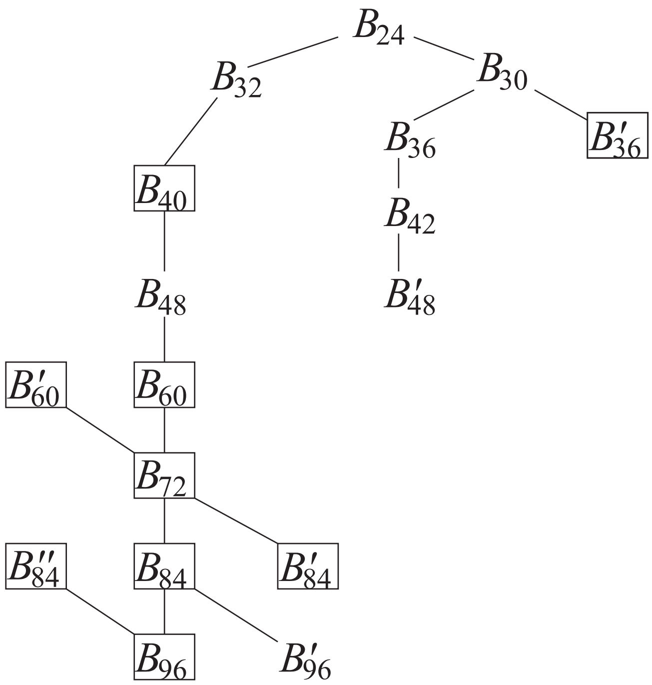

Fig. 5: The B_{24}-tribe.

We now present computational results based on the above approach. The first graph from Table 1 to consider is B_1. Set $B_{24} = B_1$ to denote the size of this graph. Then the B_{24}-tribe consists of 17 graphs and is schematically presented in Fig. 5.

After the complete B_{24}-tribe has been determined, 9 new cubic partial cubes have been found. They are squared in the figure. The chain $B_{24}, B_{30}, B_{36}, B_{42}, B'_{48}$ is the chain $B_1, B'_1, B''_1, B'''_1, B_2$ from Table 1. The graph B_{32} is the chamfered cube, B_{48} is the truncated cuboctahedron $\mathcal{CI}(Q_3)$, while B'_{96} is $\mathcal{CI}(G_4)$. The four new graphs with up to 60 vertices are depicted in Fig. 6.

The next graph from Table 1 to consider is the twisted chamfered cube. Its tribe consists of just two graphs, the second graph from the tribe is the top graph from Fig. 7. It is not isomorphic to B_{40}, hence we have one more cubic partial cube. Similarly, the $\mathcal{CI}(G_1)$-tribe consists of two graphs, the second graph has 60 vertices and is the middle graph from Fig. 7. Next, the $\mathcal{CI}(G_2)$-tribe consists of three graphs, $\mathcal{CI}(G_2)$ on 80 vertices and graphs on 70 and 60 vertices. So in this case both new graphs are obtained by a cubic contraction. The one with 60 vertices is the bottom graph from Fig. 7.

Both the $\mathcal{CI}(G_3)$-tribe and the $\mathcal{CI}(G_5)$-tribe consist of a single graph. Finally, the $\mathcal{CI}$(Icosahedron)-tribe is shown in Fig. 8, where we have set $I_{120} = \mathcal{CI}$(Icosahedron) to denote the size of this graph. The I_{120}-tribe thus brings us 8 new cubic partial cubes, the largest being on 240 vertices. It is the largest

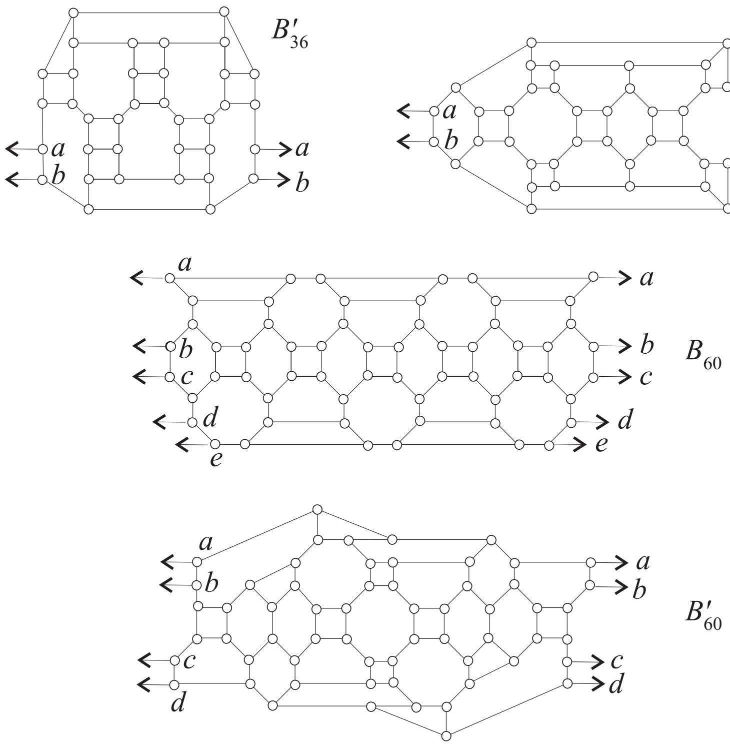

Fig. 6: Graphs from the B_{24}-tribe.

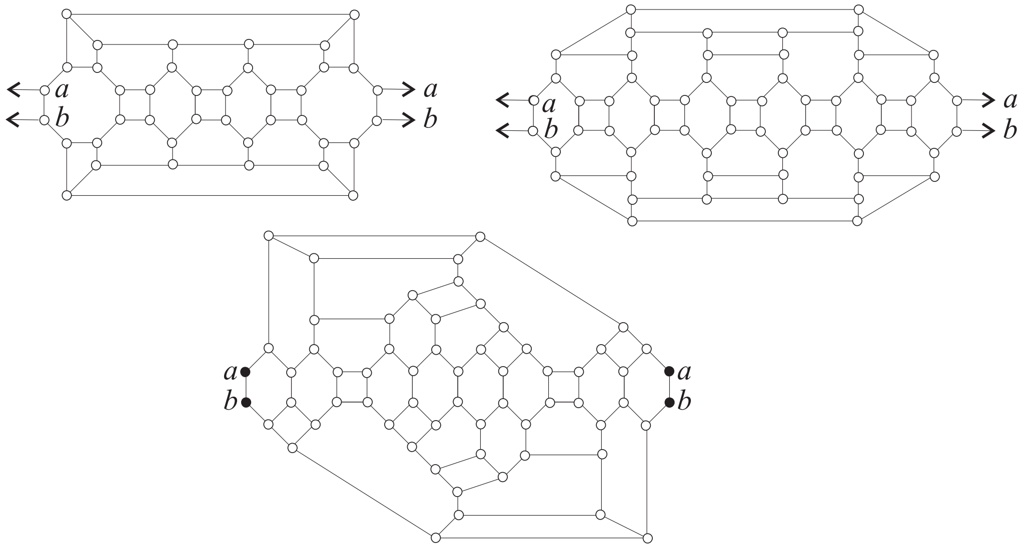

Fig. 7: The second graph from the twisted chamfered cube tribe, the second graph from the $\mathcal{CI}(G_1)$-tribe, and the graph on 60 vertices from the $\mathcal{CI}(G_2)$-tribe.

known cubic partial cube (that is not an even prism).

8 Concluding remarks

Among the known cubic partial cubes only the generalized Petersen graph $P(10,3)$ is not planar. Hence it would be interesting to find more nonplanar examples. It seems that for this sake new techniques need to be developed.

David Eppstein observed ((16)) that every simplicial line arrangement in the projective plane over the real numbers leads to a (plane) cubic partial cube. Three infinite series and 91 sporadic examples of simplicial arrangements are known (23). One of the three infinite series consists of the familiar prisms. It

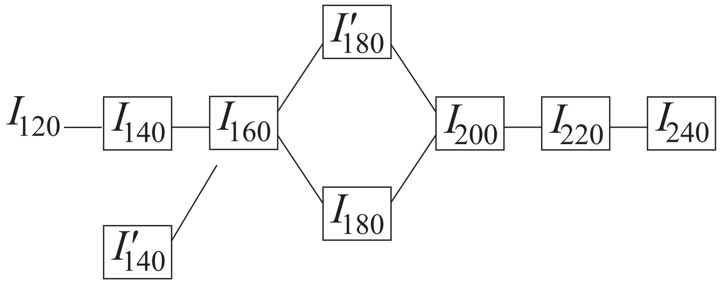

Fig. 8: The I_{120}-tribe.

would be interesting to systematically consider all other examples to see to which known or new partial cubes they lead, and what their corresponding tribes are.

References

[1] D. Avis, Hypermetric spaces and the Hamming cone, Canad. J. Math. 33 (1981) 795–802.

[2] H.-J. Bandelt, M. van de Vel, Superextensions and the depth of median graphs, J. Combin. Theory Ser. A 57 (1991) 187–202.

[3] P. Bonnington, S. Klavžar, A. Lipovec, On cubic and edge-critical isometric subgraphs of hypercubes, Australas. J. Combin. 28 (2003) 217–224.

[4] B. Brešar, S. Klavžar, A. Lipovec, B. Mohar, Cubic inflation, mirror graphs, regular maps, and partial cubes, European J. Combin. 25 (2004) 55–64.

[5] D. Bryant, V. Moulton, Neighbor-net: An agglomerative method for the construction of phylogenetic networks, Mol. Biol. Evol. 21 (2004) 255–265.

[6] V. D. Chepoi, d-Convexity and isometric subgraphs of Hamming graphs, Cybernetics 1 (1988) 6–9.

[7] H. S. M. Coxeter, Regular Polytopes, Third Edition, Dover Publications, New York, 1973.

[8] M. Deza, M. Dutour-Sikiric, S. Shpectorov, Isometric embeddings of Archimedean Wythoff polytopes into hypercubes and half-cubes, MNF Lecture Notes Series, Kyushu University, Proc. Conf. on Sphere Packings (Fukuoka 2004), 55–70.

[9] M. Deza, M. Dutour-Sikiric, S. Shpectorov, Graphs 4_n that are isometrically embeddable in hypercubes, Southeast Asian Bull. Math. 29 (2005) 469–484.

[10] M. Deza, V. P. Grishukhin, A zoo of l_1-embeddable polytopal graphs, Bull. Inst. Math. Acad. Sinica 25 (1997) 181–231.

[11] M. Deza, V. Grishukhin, M. Shtogrin, Scale-Isometric Polytopal Graphs in Hypercubes and Cubic Lattices. Polytopes in Hypercubes and $\mathbb{Z}_n$, Imperial College Press, London, 2004.

[12] M. Deza, M. Laurent, Geometry of Cuts and Metrics, Springer-Verlag, Berlin, 1997.

[13] D. Djoković, Distance preserving subgraphs of hypercubes, J. Combin. Theory Ser. B 14 (1973) 263–267.

[14] A. Dress, M. Hendy, K. Huber, V. Moulton, On the number of vertices and edges in the Buneman graph, Ann. Combin. 1 (1997) 329–337.

[15] D. Eppstein, The lattice dimension of a graph, European J. Combin. 26 (2005) 585–592.

[16] D. Eppstein, Cubic partial cubes from simplicial arrangements, Electron. J. Combin. 13 (2006) #R79, 14 pp.

[17] J.-Cl. Falmagne, S. Ovchinnikov, Media theory, Discrete Appl. Math. 121 (2002) 103–118.

[18] K. Fukuda, K. Handa, Antipodal graphs and oriented matroids, Discrete Math. 111 (1993) 245–256.

[19] The GAP Group, GAP–Groups, Algorithms, and Programming, Version 4.4; 2005 (http://www.gap-system.org)

[20] E. Gedeonova, Constructions of S-lattices, Order 7 (1990) 249–266.

[21] C. Godsil, G. Royle, Algebraic Graph Theory, Springer, New York, 2001.

[22] R. L. Graham, H. Pollak, On the addressing problem for loop switching, Bell System Tech. J. 50 (1971) 2495–2519.

[23] B. Grünbaum, Arrangements of hyperplanes, Proceedings of the Second Louisiana Conference on Combinatorics, Graph Theory and Computing (Louisiana State Univ., Baton Rouge, La., 1971), 41–106.

[24] B. Grünbaum, Arrangements and Spreads, American Mathematical Society, Providence, R.I., 1972.

[25] W. Imrich and S. Klavžar, Product Graphs: Structure and Recognition, Wiley, New York, 2000.

[26] W. Imrich, S. Klavžar, H. M. Mulder, Median graphs and triangle-free graphs, SIAM J. Discrete Math. 12 (1999) 111–118.

[27] S. Klavžar, Applications of isometric embeddings to chemical graphs, DIMACS Ser. Discr. Math. Theor. Comput. Sci. 51 (2000) 249–258.

[28] S. Klavžar, I. Gutman, B. Mohar, Labeling of benzenoid systems which reflects the vertex–distance relations, J. Chem. Inf. Comput. Sci. 35 (1995) 590–593.

[29] S. Klavžar, A. Lipovec, Partial cubes as subdivision graphs and as generalized Petersen graphs, Discrete Math. 263 (2003) 157–165.

[30] S. Klavžar, H. M. Mulder, Partial cubes and crossing graphs, SIAM J. Discrete Math. 15 (2002) 235–251.

[31] B. Mohar, C. Thomassen, Graphs on Surfaces, Johns Hopkins University Press, Baltimore, 2001.

[32] H. M. Mulder, The structure of median graphs, Discrete Math. 24 (1978) 197–204.

[33] H. M. Mulder, The Interval Function of a Graph, Math. Centre Tracts 132, Mathemtatisch Centrum, Amsterdam, 1980.

[34] S. Ovchinnikov, Hyperplane arrangements in preference modelling, J. Math. Psychology 49 (2005) 481–488.

[35] P. M. Weichsel, Distance regular subgraphs of a cube, Discrete Math. 109 (1992) 297–306.

[36] E. Wilkeit, Isometric embeddings in Hamming graphs, J. Combin. Theory Ser. B 50 (1990) 179–197.

[37] P. Winkler, Isometric embeddings in products of complete graphs, Discrete Appl. Math. 7 (1984) 221–225.